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IB Mathematics HL Revision – Step One

Chapter 1.1 – Arithmetic sequences and series; sum of finite arithmetic series; geometric sequences and series; sum of finite and infinite geometric series. Sigma notation.

Arithmetic Sequences

Definition: An arithmetic sequence is a sequence in which each term differs from the previous one by the same fixed number:

$\{u_n\}$ is arithmetic if and only if $u_{n+1} - u_n = d$.

Information Booklet

$$u_n = u_1 + (n-1)d$$

Proof/Derivation:

$$u_{n+1} - u_n = d$$
$$\therefore u_n = u_{n+1} - d$$
$$\Rightarrow u_{n+1} = u_1 + dn$$
$$\therefore u_n = u_1 + dn$$
$$\therefore u_n = u_1 + (n-1)d$$

Derivations:

$$u_1 = u_n - (n-1)d$$
$$d = \frac{u_n - u_1}{n-1}$$
$$n = \frac{u_n - u_1}{d} + 1$$

Information Booklet

$$S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$$

Proof:

$S_n = u_1 + u_2 + u_3 + \ldots + u_\text{n}$
$= u_1 + (u_1 + d) + (u_1 + 2d) + (u_1 + 3d) + \ldots + (u_1 + (n - 1)d)$
$= u_n + (u_n - d) + (u_n - 2d) + (u_n + 3d) + \ldots + (u_n - (n - 1)d)$
$\therefore 2S_n = n(u_1 + u_n)$
$$\therefore S_n = \frac{n}{2}(u_1 + u_n)$$

Derivations

$$u_n = \frac{2S_n}{n} - u_1$$

$$u_1 = \frac{2S_n}{n} - u_n$$

$$n = \frac{2S_n}{u_1 + u_n}$$

Geometric Sequences

Definition: A geometric sequence is a sequence in which each term can be obtained from the previous one by multiplying by the same non-zero constant.

{u_n} is geometric if and only if $\frac{u_{n+1}}{u_n} = r,\ n \in \mathbb{Z}^+$ where ***r*** is a constant.

Information Booklet

$$u_n = u_1 r^{n-1}$$

Proof:

$$\frac{u_{n+1}}{u_n} = r$$

$$\therefore u_n r = u_{n+1}$$

$$\therefore u_n = \frac{u_{n+1}}{r}$$

$$\rightarrow u_{n+1} = u_1 r^n$$

$$\therefore u_n = u_1 r^{n-1}$$

Derivations:

$$u_1 = \frac{u_n}{r^{n-1}}$$

$$r = \left(\frac{u_n}{u_1}\right)^{\frac{1}{n-1}}$$

$$n = \log_r \frac{u_n}{u_1} + 1 \qquad \text{(non-calculator paper)}$$

$$n = \frac{\log \frac{u_n}{u_1}}{\log r} + 1 \qquad \text{(calculator paper)}$$

Compound Interest:

$u_{n+1} = u_1 r^n$, where $u_1 =$ initial investment, $r = \frac{100\% + i\%}{100\%}$, $i =$ interest rate per compounding period, n = number of periods and $u_{n+1} =$ amount after n periods.

Information Booklet

$$S_n = \frac{u_1\left(r^n - 1\right)}{r-1} = \frac{u_1\left(1 - r^n\right)}{1-r},\ r \neq 1$$

Proof:

$S_n = u_1 + u_2 + u_3 + \ldots + u_{n-1} + u_n$
$= u_1 + u_1 r + u_1 r^2 + u_1 r^3 + \ldots + u_1 r^{n-2} + u_1 r^{n-1}$
$\therefore rS_n = (u_1 r + u_1 r^2 + u_1 r^2 + u_1 r^3 + u_1 r^4 + \ldots + u_1 r^{n-1}) + u_1 r^n$
$\therefore rS_n = (S_n - u_1) + u_1 r^n$
$\therefore rS_n - S_n = u_1 r^n - u_1$
$\therefore S_n(r-1) = u_1(r^n - 1)$
$$\therefore S_n = \frac{u_1\left(r^n - 1\right)}{(r-1)}$$

Derivations

$$u_1 = \frac{S_n(r-1)}{r^n - 1}$$

$$n = \frac{\log\left(\frac{(r-1)S_n}{u_1}\right)}{\log r}$$

$$\frac{S_n}{u_1} = \frac{r^n - 1}{r-1} = \left(1 + r + r^2 + r^3 + \ldots + r^{n-3} + r^{n-2} + r^{n-1}\right)$$

Sum to infinity

$$S_\infty = \frac{u_1}{1-r} \qquad ,|r| > 1$$

Proof:

$$S_n = \frac{u_1\left(1 - r^n\right)}{1-r},\ r \neq 1,$$
$$\Rightarrow r^\infty = 0,\ |r| < 1$$
$$\therefore S_\infty = \frac{u_1(1-0)}{1-r},\ |r| < 1$$
$$\therefore S_\infty = \frac{u_1}{1-r},\ |r| < 1$$

Sigma Notation

$$\sum_{r=1}^{n} f(n)$$

n is the number of terms, $f(n)$ is the general term and r = the first n value in the sequence. $\sum$ represents "the sum of" these the terms in this progression.

Chapter 1.2 – Exponents and logarithms. Laws of exponents; laws of logarithms. Change of base.

Definition of Exponents: $a^{\frac{x}{y}}$ means $\left(\sqrt[y]{a}\right)^x$, i.e. the numerator in an exponent is the power to which a number is raised and the denominator is the root to which it is lowered.

Laws of Exponents

$(a^x)^y = a^{xy}$

$a^x \bullet a^y = a^{x+y}$

$a^0 = 1$

$a^{-1} = \frac{1}{a}$

Definition of Logarithms: $\log_a x = y$ means $a^y = x$.

Notes:

$\ln x = \log_e x$, where e is the unique real number such that the function e^x has the same value as the slope of the tangent line, for all values of x.

$\log_a b^y = x \Leftrightarrow a^x = b^y$

$\therefore a^{\frac{x}{y}} = b \Leftrightarrow \log_a b = \frac{x}{y}$

$\therefore y \log_a b = x = \log_a b^y$

$\therefore \log_a b^y = y \log_a b$

$x = \log_a a \Leftrightarrow a^x = a$

$\therefore x = 1$

$\therefore \log_a a = 1$

$a^{\log_a x} = b \Leftrightarrow \log_a x = \log_a b$

$\therefore b = x$

$\therefore a^{\log_a b} = b$

THEREFORE $\log_a a^x = x = a^{\log_a x}$

Other Significant Equations

$a^x = e^{x \ln a}$

Proof:

$$e^{x\ln a} = e^{\ln a^x}$$
$$= a^x$$
$$\therefore a^x = e^{x\ln a}$$

Other Significant Equations

$$\log_b a = \frac{\log_c a}{\log_c b}$$

Proof:

$$\log_b a = x$$
$$\therefore b^x = a$$
$$\therefore \log_c b^x = \log_c a$$
$$\therefore x\log_c b = \log_c a$$
$$\therefore x = \frac{\log_c a}{\log_c b}$$
$$\therefore \log_b a = \frac{\log_c a}{\log_c b}$$

Laws of Logarithms

$$\log_n ab = \log_n a + \log_n b$$

Proof:

$$\log_n a + \log_n b = x$$
$$\therefore \log_n a = x - \log_n b$$
$$\therefore n^{x-\log_n b} = a$$
$$\therefore \frac{n^x}{n^{\log_n b}} = a$$
$$\therefore \frac{n^x}{b} = a$$
$$\therefore n^x = ab$$
$$\therefore \log_n n^x = \log_n ab$$
$$\therefore x = \log_n ab$$
$$\therefore \log_n ab = \log_n a + \log_n b$$

Laws of Logarithms

$$\log_n ab = \log_n a + \log_n b$$

Proof:

$$\log_n a - \log_n b = x$$
$$\therefore \log_n a = x + \log_n b$$
$$\therefore n^{x+\log_n b} = a$$
$$\therefore n^x \bullet n^{\log_n b} = a$$
$$\therefore bn^x = a$$
$$\therefore n^x = \frac{a}{b}$$
$$\therefore \log_n n^x = \log_n \frac{a}{b}$$
$$\therefore x = \log_n \frac{a}{b}$$
$$\therefore \log_n \frac{a}{b} = \log_n a - \log_n b$$

Chapter 1.3 – Counting principles, including permutations and combinations. The binomial theorem: expansion of $(a+b)^n$, $n \in \mathbb{N}$.

The Product Principle

If there are *m* different ways of performing an operation and for each of these there are *n* different ways of performing a second **independent** operation, then there are *mn* different ways of performing the two operations in succession.

The number of different ways of performing an operation is equal to the sum of the different **mutually exclusive** possibilities.

Factorial Notation

$n!$ is the product of the first *n* positive integers for $n \geq 1$.
$$n! = n(n-1)! = n(n-1)(n-2)! = n(n-1)(n-2)(n-3)! = etc$$
$$\therefore 1! = 1(1-1)! = 1 \times 0!$$
$$\therefore 0! = 1$$

Permutations (in a line)

A **permutation** of a group of symbols is *any arrangement* of those symbols in a definite *order*.

Explanation: Assume you have *n* different symbols and therefore *n* places to fill in your arrangement. For the first place, there are *n* different possibilities. For the second place, no matter what was put in the first place, there are $n-1$ possible symbols to place, for the r^{th} place there are $n-r+1$ possible places until the point where $r = n$, at which point we have saturated all the places. According to the product principle, therefore, we have $n \times (n-1) \times (n-2) \times (n-3) \times ... \times 1$ different arrangements, or $n!$

If symbols are fixed in place, revert to the product principle. Since there is only one possibility for whichever place the symbol(s) is fixed at, the number of possibilities is equal to $\frac{n!}{n-r+1}$ where r is the place at which that symbol is fixed.

<u>Permutations</u> (in a circle)

The best way to think of permutations in a circle is as permutations in a line where you have to divide the normal total number of possibilities for permutations in a line by the number of different identical positions the symbols can have where they have simply shifted to the right by one place. Logically, there are *n* different positions where this is the case thus the number of possibilities is equal to $\frac{n!}{n} = (n-1)!$.

<u>Combinations</u>

A combination is a selection of objects *without* regard to order or arrangement.

$_nC_r = C_r^n = {}^nC_r = \binom{n}{r}$ is the number of combinations on *n* distinct symbols taken *r* at a time.

Since the combination does not take into account the order, we have to divide the permutation of the total number of symbols available by the number of redundant possibilities. Since we are choosing a particular number of symbols *r*, these symbols have a number of redundancies equal to the permutation of the symbols (since order doesn't matter). However, we also need to divide the permutation *n*! by the permutation of the symbols that are *not* selected, that is to say $n-r$.

$$\frac{\left(\frac{n!}{r!}\right)}{(n-r)!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

<u>Binomial Expansion</u>

Taking each term in the expansion of $(a+b)^n,\ n \in \mathbb{N}$ to be a symbol, it can be seen that the coefficient for each symbol, which is a different value of $a^{n-r}b^r$, (*n* being the exponent and *r* being the power to which be is raised in this particular term) is determined by the number of different possible arrangements containing *n*−*r a* symbols and *r b* symbols. Thus, the coefficient for any symbol $a^n b^{n-r}$ is equal to $\binom{n}{r}$. Since the expansion of $(a+b)^n,\ n \in \mathbb{N}$ is effectively the sum of all the symbols and their coefficients, we are left with the result that $(a+b)^n,\ n \in \mathbb{N} = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r$ and the value of each term is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$.

The **constant term** is the term containing no variables, often b^r. See H&H p.215 example 18.

When finding the coefficient of x^n, always consider the set of all terms containing x^n (see H&H p.215 example 19).

Chapter 1.4 – Proof by mathematical induction. Forming conjectures to be proved by mathematical induction.

Ninety percent of the points for mathematical induction questions can be obtained simply by using the correct form, so it is very important to memorise the two forms of mathematical induction and lay the proof out accordingly.

The Principle of Mathematical Induction

Suppose P_n is a proposition which is defined for every integer $n \geq a$, $a \in \mathbb{Z}$. Now if P_a is true, and P_{k+1} is true whenever P_k is true, then P_n is true for all $n \geq a$.

Sums of Series

First step: Prove that P_1 is true if proving for all $n \in \mathbb{Z}^+$, or that P_0 is true if proving for $n \in \mathbb{N}$ (including 0).

Second step: Assume that P_k is true, and state the consequences of this assumption.

Third step: Using the assumption, manipulate your equation (the general equation for k terms + the $(k+1)^{th}$ term) to give you the general equation where the variable k has been replaced by $k+1$ wherever it appears.

Final step, state: Thus P_{k+1} is true whenever P_k is true. Since P_1 is true, P_n is true for all [aforementioned set of numbers].

Note: **Always look for common factors.**

Divisibility

Prove that $f(n)$ is divisible by s for $n \in \mathbb{Z}$, $n \geq r$

First step: Prove that P_r is true where r is your lower limit for which P_k is true.

Second Step: Assume that P_k is true: $f(k) = sA$ where A is an integer.

Third Step: Separate out an a^k, $a \in \mathbb{Z}$ from $f(k+1)$ and substitute the term out for the value of a^k solved from $f(k)$.

Fourth step: Express $f(k+1)$ as a product of s.

Final step, state: Thus $f(k+1)$ is divisible by s if $f(k)$ is divisible by s. Hence, P_{k+1} is true whenever P_k is true and P_r is true, $\therefore P_n$ is true.

Chapter 1.5 – Complex number $\mathrm{i}=\sqrt{-1}$; the terms real part, imaginary part, conjugate, modulus and argument. Cartesian form $z=a+\mathrm{i}b$. Modulus-argument form $z=r(\cos\theta+\mathrm{i}\sin\theta)=r\mathrm{e}^{\mathrm{i}\theta}=r\operatorname{cis}\theta$. The complex plane, or Argand diagram.

Cartesian form

In Cartesian form, all complex numbers z are written in the form a + ib, $a,b\in\mathbb{R}$. a contains no imaginary component and as such is known as the *real part* whereas b is a product of the imaginary unit i, and thus is known as the *imaginary part*. It is important to note that real numbers are merely complex numbers with $b=0$ and imaginary numbers are merely complex numbers with $a=0$, thus the category real numbers is a subset of complex numbers.

Equality of Complex Numbers

Two complex numbers are equal when their real parts are equal and their imaginary parts are equal, i.e. if $a+b\mathrm{i}=c+d\mathrm{i}$, then $a=c$ and $b=d$.

Proof:
Assume $b\neq d$.

$a+b\mathrm{i}=c+d\mathrm{i},\ a,\ b,\ c,\ d\in\mathbb{R}$

$\therefore b\mathrm{i}-d\mathrm{i}=c-a$

$\therefore (b-d)\mathrm{i}=c-a$

$\therefore \mathrm{i}=\dfrac{c-a}{b-d}$

Therefore the statement $b\neq d$ must be false since i is imaginary and $\dfrac{c-a}{b-d}$ is real, therefore $b=d\ \therefore a=c$.

Conjugates

The conjugate of a complex number $z=a+\mathrm{i}b$, is $a-\mathrm{i}b=z^*$. In other words, complex conjugates are complex numbers z where the sign of the imaginary part is inverted.

Properties:

$(z^*)^*=z$

$(z_1+z_2)^*=z_1^*+z_2$ and $(z_1-z_2)^*=z_1-z_2$

$(z_1z_2)^*=z_1^*\times z_2^*$ and $\left(\dfrac{z_1}{z_2}\right)^*=\dfrac{z_1^*}{z_2^*},\ z_2\neq 0$

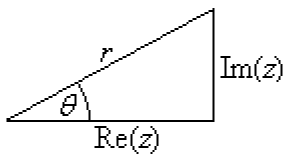
$(z^n)^*=(z^*)^n,\ n\in\mathbb{N},\ 1\leq n\leq 3$

$z+z^*$ and zz^* are real.

Polar Form

Complex numbers can, however, be expressed on an Argand diagram. An Argand diagram is like a Cartesian diagram where the *y*-axis values represent the coefficient of the imaginary part of a particular complex number and the *x*-axis values represent the value of the real part of that number. Thus, complex numbers can be represented on points on this diagram, where, as stated above, its position relative to the *x*-axis determines the value of its real part and its position relative to the *y*-axis determines the coefficient of its imaginary part.

Expressing the complex number on an Argand diagram allows us to express it in terms of its position relative to the origin: with an argument, or angle θ (in radians) relative to the *x*-axis in the positive direction (going anti-clockwise) and a modulus *r*, or length of the straight line drawn between the point and the origin. We can thus find the value of the real part and imaginary part of the complex number in polar form:



As is demonstrated above, the real part can be said to be equal to $r\cos\theta$ and the coefficient of the imaginary part can be said to be equal to $r\sin\theta$. Thus, the complex number z can be expressed as $z = r\cos\theta + \mathrm{i}r\sin\theta = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$. This is known as the polar, or modulus-argument form.

Note: The modulus *r* of a complex number is the magnitude of its unit vector on an Argand diagram. As such, it can also be written as $|z|$ and is equal to $\sqrt{a+b}$ (from the complex number as expressed in Cartesian form).

Note: The argument of any complex number z can be expressed as $\arg z$ rather than θ.

Notes (all these are easily proven so proofs shall not be made):

$$|z^*| = |z|$$

$$|z^2| = zz^*$$

$$|z_1 z_2| = |z_1||z_2| \text{ and } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$$

$$|z_1 z_2 z_3 \ldots . z_n| = |z_1||z_2||z_3|\ldots|z_n| \text{ and } |z^n| = |z|^n,\ n \in \mathbb{Z}^+$$

Notes (Proofs may potentially be provided at a later date…):

$$\mathrm{cis}\theta \times cis\phi = \mathrm{cis}(\theta + \phi)$$

$$\frac{\mathrm{cis}\theta}{\mathrm{cis}\phi} = \mathrm{cis}(\theta - \phi)$$

$$\mathrm{cis}(\theta + \mathrm{k}2\pi) = \mathrm{cis}\theta,\ k \in \mathbb{Z}$$

Converting between Cartesian and Polar Form

$z = a + bi = r\cos\theta + ir\sin\theta$

$\therefore a = r\cos\theta, b = r\sin\theta$

$\Rightarrow \theta = \arctan\frac{b}{a}$ (from Argand diagram)

$\Rightarrow r = \sqrt{a+b}$ (from Argand diagram)

Chapter 1.6 – Sums, products and quotients of complex numbers.

Operations with complex numbers are identical to those for radicals.

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$
$$(a+bi)-(c+di)=(a-c)+(b-d)i$$
$$(a+bi)(c+di)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i$$
$$\frac{a+bi}{c+di}=\left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)=\left(\frac{ac-adi+bci-bdi^2}{c^2+d^2}\right)=\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i$$

Chapter 1.7 – De Moivre’s theorem. Powers and roots of a complex number.

De Moivre’s theorem states that for any complex number *z*:

$z^n = r^n \text{cis} n\theta$

Notes:

$a+bi = r\text{cis}\theta$

$\arg z = \theta$

$\arg z^n = n\theta$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta - \phi)$

$z_1 \times z_2 = r_1 r_2 \text{cis}(\theta + \phi)$

$\theta = \arctan\frac{b}{a}$

Proof by mathematical induction for De Moivre’s Theorem:

Required to prove that: $(|z|\text{cis}\theta)^n = |z|^n \text{cis}\, n\theta$

P_1: If $n=1$, then $(|z|\text{cis}\theta)^1 = |z|^n \text{cis}\, 1(\theta) \therefore P_1$ is true.

Assume P_k is true $\therefore (|z|\text{cis}\,\theta)^k = |z|^k \text{cis}\, k\theta$

$$P_{k+1}: \begin{aligned} &(|z|\text{cis}\theta)^k \times |z|\text{cis}\theta \\ &= |z|^k \text{cis}(k\theta) \times |z|\text{cis}\theta \\ &= |z|^{k+1}\text{cis}(k\theta+\theta) \\ &= |z|^{k+1}\text{cis}(k+1)\theta \end{aligned}$$

Thus P_{k+1} is true whenever P_k is true and P_1 is true $\therefore P_n$ is true.

Roots of Complex Numbers

The nth roots of complex number c are the n solutions of $z^n = c$.

Two methods of solving: factorisation and the nth root method.

nth root method:

$$z^n = c$$
$$\therefore z^n = r\text{cis}(\theta + 2k\pi)$$
$$\therefore z = [r\text{cis}(\theta + 2k\pi)]^{\frac{1}{n}}$$
$$\therefore z = r^{\frac{1}{n}}\text{cis}\left(\frac{\theta + 2k\pi}{n}\right), \text{k} = 0,1,\ldots,n\text{-}1$$

Note: the nth roots of unity are the solutions of $z^n = 1$

Chapter 1.8 – Conjugate roots of polynomial equations with real coefficients.

Real Polynomials

A real polynomial is a polynomial with only real coefficients:

Polynomials	Degree	Name
$ax + b,\ a \neq 0$	1	Linear
$ax^2 + bx + c,\ a \neq 0$	2	Quadratic
$ax^3 + bx^2 + cx + d,\ a \neq 0$	3	Cubic
$ax^3 + bx^2 + cx + d,\ a \neq 0$	4	Quartic
$k_1x^n + k_2x^{n-1} + k_3x^{n-2} + \ldots. + k_{n-1}x + k_n,\ k_1 \neq 0$	n	(General)

$a, b, c, d, e, k \in \mathbb{R}$

a or k_1 is the leading coefficient and the term not containing the variable x is the constant term (as stated above).

Polynomial Multiplication

Every term in the first polynomial must be multiplied by every term in the other.

Algorithm: Synthetic multiplication – detach coefficients and multiply as in ordinary multiplication of large numbers

Example: $(ax^3 + bx^2 + cx + d)(ex + f)$

	a	b	c	d
		$\times$	e	f
	af	bf	cf	df
ae	be	ce	de	0
ae	$af + be$	$bf + ce$	$cf + de$	df
x^4	x^3	x^2	x^1	x^0

$$\therefore (ax^3 + bx^2 + cx + d)(ex + f) = aex^4 + (af + be)x^3 + (bf + ce)x^2 + (cf + de)x + df$$

Polynomial Division

Division Algorithm: H&H p.170 LEARN (it's hell to type out on word)

If $P(x)$ is divided by $ax+b$ until a constant remainder R is obtained:

$\frac{P(x)}{ax+b} = Q(x) + \frac{R}{ax+b}$ where $ax+b$ is the divisor, $P(x)$ is the polynomial, $Q(x)$ is the quotient and R is the remainder.

Derivations:

$$P(x) = (ax+b)Q(x) + R$$
$$Q(x) = \frac{P(x)-R}{ax+b}$$
$$R = P(x) - (ax+b)Q(x)$$
$$ax+b = \frac{P(x)-R}{Q(x)}$$

Roots and Zeros

A zero of a polynomial is a value of the variable which makes the polynomial equal to zero (*x*-axis intercept). The roots of a polynomial equation are values of the variable which satisfy the equation in question.

The roots of $P(x)=0$ are the zeros of $P(x)$.

In polynomials of the form:
$P(x) = (x-\alpha)Q(x)$ the roots of the equation occur at $\alpha = 0$ and the solutions of $Q(x)$. When finding the roots of polynomials, it is important to factorise $Q(x)$ until it is a product of linear factors and quadratic factors.

Linear factors of quadratics:

$$ax^2+bx+c = 0 = (x-\alpha)(x-\beta)$$
$$\therefore x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b+\sqrt{b^2-4ac}}{2a} = \alpha = \frac{-b-\sqrt{b^2-4ac}}{2a} = \beta$$
$$\therefore ax^2+bx+c = \left(x + \frac{b-\sqrt{b^2-4ac}}{2a}\right)\left(x + \frac{b+\sqrt{b^2-4ac}}{2a}\right)$$

If $z = \frac{b+\sqrt{b^2-4ac}}{2a}$, $ax^2+bx+c = (x+z)(x+z^*) = 0$, $z \in \mathbb{C}$
z and z^* are conjugate roots of $ax^2+bx+c=0$

For polynomials of even degree, every root has a conjugate root, for polynomials of odd degree, every root except one has a conjugate root.

Every polynomial of degree n has n roots, but where there is a factor $x-0$, the polynomial has repeated roots.

Chapter 2.1 – Concept of function $f:x \mapsto f(x)$: domain, range; image (value). Composite functions $f \circ g$; identity function. Inverse function f^{-1}.

Algebraic Test: If a relation is given as an equation, and the substitution of any value for *x* results in one and only one value of *y*, we have a function. (Note that the algebraic test can be used as a definition for what a function is).

Geometric Test: If at any point along the *x*-axis, there are two *y*-axis values, the graph is not a function.

In composite functions, the right-most function is a function of *x* and those that aren't are merely functions of the functions the right of it. That is to say that if you define a function $f(x)$ as a relationship given as an equation where the substitution of any value for *x* results in one and only one value of *y*, the definition of the function $g \circ f : x \mapsto g(f(x))$ is a relationship given as an equation where the substitution for any value for $f(x)$ results in one and only one value of *y*.

The domain of a relation is the set of permissible values that *x* may have.
The range of a relation is the set of permissible values that *y* may have.

Note: The stated domain and range of the relation in question must be applied before it is determined whether or not this relation constitutes a function.

<u>Interval Notation</u>

$$x \geq a \rightarrow \{x : x \geq a\} \Leftrightarrow x \in [a, \infty[$$
$$y \geq a \rightarrow \{y : y \geq a\} \Leftrightarrow y \in [a, \infty[$$

$$x < a \rightarrow \{x : x < a\} \Leftrightarrow x \in]-\infty, a[$$
$$y < a \rightarrow \{y : y < a\} \Leftrightarrow y \in]-\infty, a[$$

$$\{x : x \in \mathbb{R}\} \Leftrightarrow x \in \mathbb{R}$$
$$\{y : y \in \mathbb{R}\} \Leftrightarrow y \in \mathbb{R}$$

$$a < x \leq b \rightarrow \{x : -2 < x \leq b\} \Leftrightarrow x \in]a, b]$$
$$a < y \leq b \rightarrow \{y : -2 < y \leq b\} \Leftrightarrow y \in]a, b]$$

[*a* means that *a* is the lower limit and that *x* can equal *a*
]*a* means that *a* is the upper limit and that *x* can't equal *a*

b] means that *b* is the upper limit and that *x* can equal *b*
b[means that *b* is the upper limit and that *x* can't equal *b*.

$]-\infty, \infty[$ where there is no limit on one side, ∞ must be excluded in the notation.

Inverse functions

Where a function $f(x)$ substitutes x values for a different value, the inverse function $f^{-1}(x)$ is the function that substitutes $f(x)$ values for x values. Graphically, the inverse of a function is that function reflected in the line $y = x$.

As a direct result of this, any function in which more than one x value is substituted to the same value of y (called a many-to-one function) has no inverse. This is represented graphically by the horizontal bar test, where if you can draw a line parallel to the x-axis that crosses the function more than once, the function has no inverse (if the converse is true, then the function is a one-to-one function). Otherwise stated, one-to-one functions have an inverse whereas many-to-one functions do not. The domain must be taken into account when categorising a function as one-to-one rather than many-to-one, for example $f(x) = \sin x$ appears to be a many-to-one function, but if the domain is $\left[0, \frac{\pi}{2}\right]$ then the function is a one-to-one function and has an inverse.

Chapter 2.2 – The graph of a function; its equation $y = f(x)$. Function graphing skills: use of a GDC to graph a variety of functions, investigation of key features of graphs, solutions of equations graphically.

When functions are graphed, the function $f(x)$ is always represented on the y axis.

The asymptote is an x or y value for which there is defined value of the function, generally appearing where the denominator in the function has gone to 0.

The roots or equations (or zeros of functions) can be found graphically by noting the points at which the function crosses the x axis. For the solution to two equations represented as functions as $f(x)$ and $g(x)$ respectively where $f(x) = g(x)$, the solution(s) can be found at the intercept(s) of the two functions.

Chapter 2.3 – Transformations of graphs: translations; stretches; reflections in the axes. The graph of $y = f^{-1}(x)$ as the reflection in the line $y = x$ of the graph of $y = f(x)$. The graph of $y = \frac{1}{f(x)}$ from $y = f(x)$. The graphs of the absolute value functions, $y = |f(x)|$ and $y = f(|x|)$.

Transformations:

$y = f(x) + b$ translates the graph b units in the positive y direction.
$y = f(x - a)$ translates the graph a units in the positive x direction.

$y = pf(x)$ stretches the graph parallel to the y-axis with factor p.
$y = f\left(\frac{x}{q}\right)$ stretches the graph parallel to the x-axis with factor q.

$y = -f(x)$ reflects the graph in the x-axis.
$y = f(-x)$ reflects the graph in the y-axis.

If given $f(x)$ and required to graph $\frac{1}{f(x)}$, realise where the graph of $f(x)$ gets steeper, the graph of $\frac{1}{f(x)}$ gets flatter, and that the greater the magnitude of $f(x)$ at any point x, the magnitude of $f(x)$ is lesser. However, where $f(x)$ is less than 1 and tends towards 0, the slope gets steeper until it reaches the asymptote at $f(x) = 0$.

$y = |f(x)|$ The domain where $f(x) < 0$ is reflected in the x-axis, all else is unchanged.
$y = f(|x|)$ The domain $x > 0$ is taken and the graph is reflected in the y-axis.

Properties of $|x|, x \in R$
$|x|$ is the distance from 0 to x on the number line
$|x| = \left(\sqrt{x}\right)^2 \Rightarrow |x|^2 = x^2$
$|x| \geq 0$
$|x| \geq x$
$|-x| = |x|$
$|xy| = |x||y| \Rightarrow \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
$|x^n| = |x|^n, n \in Z$
$|x + y| \leq |x| + |y|$
$|x - y| \geq |x| - |y|$
$|x - a|$ is the difference between x and a on the real number line
$|x - y| = |y - x|$

Chapter 2.4 – The reciprocal function $x \mapsto \frac{1}{x}, \ x \neq 0$: its self-inverse nature.

Since the function that maps $\frac{1}{x}$ onto x is known as the inverse of $x \mapsto \frac{1}{x}, \ x \neq 0$ and is $\frac{1}{x}$, the reciprocal function is effectively its own inverse.

Chapter 2.5 – The quadratic function $x \mapsto ax^2 + bx + c$: its graph. Axis of symmetry $x = -\frac{b}{2a}$. The form $x \mapsto a(x-h)^2 + k$. The form $x \mapsto a(x-p)(x-q)$.

All quadratic functions can be written in the form $x \mapsto ax^2 + bx + c$. However, some can be written in the form $x \mapsto a(x-p)(x-q), \ a \neq 0$ while others can also be written in the form. $x \mapsto a(x-h)^2 + k$. If $k > 0$, the quadratic can only be written in the latter form (short of using complex roots) however if $k \le 0$, the quadratic can be expressed in both forms.

The first form is beneficial for finding the roots of the quadratic, since they are equal to *p* and *q*. The second, however, is much more useful for finding the axis of symmetry (*h*) of the quadratic and determining where the minimum (or maximum for $a < 0$) point (*k*) of the quadratic is.

Thanks to the properties of the second equation, putting the general equation for $x \mapsto ax^2 + bx + c$ in the form $x \mapsto a(x-h)^2 + k$ allows us to find the general equation for the position of the axis of symmetry and the position of the min/max point on the graph.

Completing the square:

$$ax^2 + bx + c = a\left(x^2 + \frac{bx}{a}\right) + c$$

$$= a\left(x^2 + 2x\left(\frac{b}{2a}\right) + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c$$

$$= a\left(x^2 + 2x\left(\frac{b}{2a}\right) + \left(\frac{b}{2a}\right)^2\right) - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = a(x-h)^2 + k$$

$$\therefore \frac{b}{2a} = -h, \quad -\frac{b^2}{4a} + c = k$$

$$\therefore -\frac{b}{2a} = h$$

Chapter 2.6 –The solution of $ax^2 + bx + c = 0, \ a \ne 0$. The quadratic formula. Use of the discriminant $\Delta = b^2 - 4ac$.

Completing the square also allows us to find the general equation for the roots of an equation.

$$ax^2 + bx + c = 0$$

$$\Rightarrow ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$\therefore a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0$$

$$\therefore a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This equation is known as the quadratic formula.

What is important to note in this formula is the discriminant $\Delta = b^2 - 4ac$. This determines whether or not the quadratic has a real solution.

Where $\Delta = 0$, the quadratic has two repeated roots, where $\Delta > 0$, the quadratic has two different roots and where $\Delta < 0$ the quadratic has no real roots.

Chapter 2.7 – The function: $x \mapsto a^x$, $a > 0$. The inverse function $x \mapsto \log_a x$, $x > 0$. Graphs of $y = a^x$ and $y = \log_a x$. Solutions of $a^x = b$ using logarithms.

The function $x \mapsto a^x$ has a curvature that tends towards zero as x tends towards infinity. The curvature also tends towards zero as x tends towards negative infinity. The gradient, however, constantly increases from 0 at $x = -\infty$ to $\frac{1}{0}$ at $x = \infty$. Its inverse function is merely a reflection of this in the x-axis.

We can find the solutions of $a^x = b$ using logarithms:

$$a^x = b$$

$$\therefore \log_a a^x = \log_a b$$

$$\therefore x = \log_a b$$

For harder equations, the solutions can be found the following way:

$$a^x = b$$
$$\therefore \log a^x = \log b$$
$$\therefore x = \frac{\log b}{\log a}$$

Chapter 2.8 – The exponential function $x \mapsto e^x$. The logarithmic function $x \mapsto \ln x$, $x > 0$.

It is known from topic 1.1 that the equation for compound interest is:

$u_{n+1} = u_1 r^n$, where $u_1 =$ initial investment, $r = \frac{100\% + i\%}{100\%}$, $i =$ interest rate per compounding period, n = number of periods and $u_{n+1} =$ amount after n periods.

It is clear that $r = 1 + i$, and that if we treat the initial investment to be the 0th term, we get an equation where $u_n = u_0 r^n = u_0 (1+i)^n$.

If r is (instead) the percentage rate per year, t the number of years and N the number of interest payments per year, then $u_n = u_0 \left(1 + \frac{r}{N}\right)^{Nt} = u_0 \left(1 + \frac{1}{\left(\frac{N}{r}\right)}\right)^{\frac{N}{r} \times rt}$

Let $\frac{N}{r} = a$, then $u_n = u_0 \left(1 + \frac{1}{a}\right)^{a \times rt} = u_0 \left[\left(1 + \frac{1}{a}\right)^a\right]^{rt}$

But $a^x = e^{x \ln a}$ where a and x are any real number, thus:

$$u_n = u_0 \left[\left(1 + \frac{1}{a}\right)^a\right]^{rt}$$

$$\Rightarrow \lim_{a \to \infty} \left(1 + \frac{1}{a}\right)^a = \mathrm{e}$$

$\therefore u_n \cong u_0 e^{rt}$ for large values of a.

Growth and decay works in a similar way, putting the sequence in the form $u_n = u_0 f\left(\left(1 + \frac{1}{a}\right)^a\right)$.

Chapter 2.9 – Inequalities in one variable, using their graphical representation. Solution of $g(x) \geq f(x)$, where f, g are linear or quadratic.

Inequalities can be solved using a sign diagram, where it is key to remember that the arrow represents all the values for which $f(x) > 0$, regardless of whether we're trying to find the values for which the overall function is greater than or less than x.

Inequality laws:

$a > b \Rightarrow a + c > b + c$

$a > b,\ c > 0 \ \Rightarrow\ ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

$a > b,\ c < 0 \ \Rightarrow\ ac < bc$ and $\frac{a}{c} < \frac{b}{c}$.

$a > b \geq 0 \Rightarrow a^2 > b^2$.

Sign Diagram Notes:

The horizontal line of a sign diagram corresponds to the x-axis.
The critical values are values of x when the function is zero or undefined.
A positive sign (+) corresponds to the fact that the graph is above the x-axis.
A negative sign (−) corresponds to the fact that the graph is below the x-axis.
When a factor has an odd power there is a change of sign about that critical value. When a factor has an even power there is no sign change about that critical value.
For a quadratic factor $ax^2 + bx + c$ where $\Delta = b^2 - 4ac < 0$;

$ax^2 + bx + c > 0, x \in R, a > 0,$

$ax^2 + bx + c < 0, x \in R, a < 0.$

There is no critical value in either case.

Solving inequalities: Procedure

Make the RHS = 0 by transferring all terms to LHS.
Fully factorise the LHS.
Draw a sign diagram of the LHS.
Solve.

Note: Do not, under any circumstances, cross-multiply. This removes certain terms from the equations and prevents one from finding all the solutions.

Chapter 2.10 – Polynomial functions. The factor and remainder theorems, with application to the solution of polynomial equations and inequalities.

It is important to note that the higher the order of the polynomial function, the steeper the line for $|x| > 1$ and the less steep the line for $|x| < 1$. Odd orders have negative values where $x < 0$ but even order always having positive values unless translated down.

Repeated roots merge on a graph to appear as only one root. Repeated roots signify a point of inflection.

The factor theorem

According to polynomial division,

$P(x) = Q(x)(x - k) + R$

Thus, where $x = k$

$P(k) = Q(k) \times 0 + R$

$\therefore P(k) = R$

Therefore when polynomial $P(x)$ is divided by $x-k$ until a constant remainder R is obtained then $R = P(k)$

The remainder theorem

If k is a zero of $P(x)$ thus:

$P(x) \Leftrightarrow P(k) = 0$

$\Leftrightarrow R = 0$

$\Leftrightarrow P(x) = +(x)(x-k)$

$\Leftrightarrow (x-k)$ is a factor of $P(x)$

In general: k is a zero of $P(x) \Leftrightarrow (x-k)$ is a factor of $P(x)$.

Using the remainder theorem to find the remainder allows us to find how far the minimum/maximum point of the graph of $\frac{P(x)}{Q(x)} - R = 0$ is shifted away from the x-axis.

Using the factor theorem allows us to quickly find the roots (if the factor is known) or the factors (if the root is known) of the polynomial equation and thus solve polynomial equations and inequalities quickly.

Chapter 3.1 – The circle: radian measure of angles; length of an arc; area of a sector.

One radian is defined as the angle that subtends an arc of length equal to the radius. Thus,

$$\frac{\theta_{arc}}{\theta_{total}} = \frac{r}{C}$$

$$\Rightarrow C = 2\pi r$$

$$\therefore \frac{\theta_{arc}}{\theta_{total}} = \frac{r}{2\pi r} = \frac{1}{2\pi}$$

Thus 1 radian is $\frac{1}{2\pi}$ of the angle of a full circle, thus there are 2π radians per full turn.

This means that: $2\pi^c = 360°$, so $1^c = \frac{180°}{\pi}$ and $1° = \frac{\pi^c}{180}$.

Thus, the length of an arc in radians is:

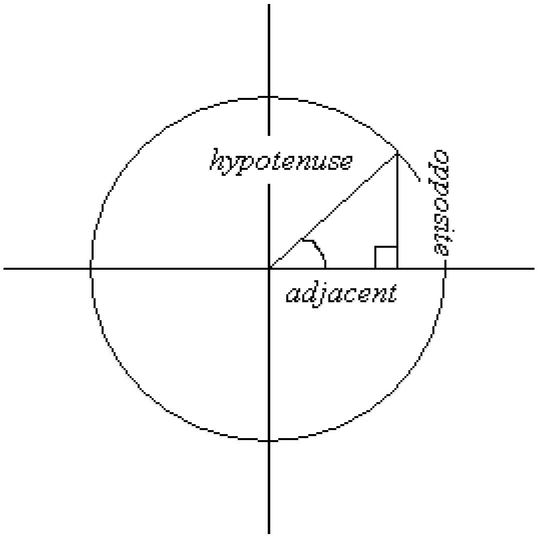
$$l = \frac{\theta}{2\pi} \times 2\pi r = \theta r$$

And the area of a sector is:

$$A = \frac{\theta}{2\pi} \times \pi r^2 = \frac{\theta r^2}{2}$$

Chapter 3.2 – Definition of $\cos\theta$ and $\sin\theta$ in terms of the unit circle. Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$. Definition of $\sec\theta$, $\csc\theta$ and $\cot\theta$. Pythagorean identities: $\cos^2\theta + \sin^2 = 1$; $1 + \tan^2\theta = \sec^2\theta$; $1 + \cot^2\theta = \csc^2\theta$.

The Sine Function



The sine function of θ means that for a right angled triangle with angle θ (as shown in the diagram), the ratio of the length of the opposite side to that of the hypotenuse is equal to the sine function of θ.

Pythagoras's theorem states that $a^2 + b^2 = c^2$ where a is the opposite side, b is the adjacent side and c is the hypotenuse, thus we can state that:

$$\sin\theta = \frac{a}{c}$$

$$\cos\theta = \frac{b}{c}$$

$$\tan\theta = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{\sin\theta}{\cos\theta}$$

The unit circle has radius 1. This means that the hypotenuse has radius 1. We can thus use Pythagoras's Theorem to state that $a^2 + b^2 = 1$ and that therefore $a = \pm\sqrt{1-b^2}$ and $b = \sqrt{1-a^2}$.

Thus, in terms of the unit circle,

$$\cos\theta = \frac{b}{c} = \frac{\sqrt{1-a^2}}{1} = \sqrt{1-a^2} = b$$

$$\sin\theta = \frac{a}{c} = \frac{\sqrt{1-b^2}}{1} = a = \sqrt{1-b^2}$$

And

$$\tan\theta = \frac{a}{b} = \sqrt{\frac{1-b^2}{1-a^2}} = \frac{a}{\sqrt{1-a^2}} = \sqrt{\frac{b}{1-b^2}}$$

This tells us the variance of each function in terms of the unit length of the opposite and in terms of the unit length of the adjacent. In the plotting the sine curve, we plot a on the y-axis and θ on the x-axis.

Thus we obtain:

$$a = \sin\theta$$
$$\sqrt{1-a^2} = \cos\theta$$
$$\therefore 1-a^2 = \cos^2\theta$$
$$\therefore a^2 = 1-\cos^2\theta = \sin^2\theta$$
$$\therefore \sin^2\theta + \cos^2\theta = 1$$

And

$$\tan\theta = \frac{a}{\sqrt{1-a^2}}$$
$$\therefore \tan^2\theta = \frac{a^2}{1-a^2}$$
$$\Rightarrow a = \sin\theta$$
$$\therefore \tan^2\theta = \frac{\sin^2\theta}{1-\sin^2\theta}$$
$$\therefore 1+\tan^2\theta = 1+\frac{\sin^2\theta}{1-\sin^2\theta} = \frac{\left(1-\sin^2\theta\right)+\sin^2\theta}{1-\sin^2\theta}$$
$$\Rightarrow 1-\sin^2\theta = \cos^2\theta$$
$$\therefore 1+\tan^2\theta = \frac{1}{\cos^2\theta} = \sec^2\theta$$

Also

$$\tan^2\theta = \frac{\sin^2\theta}{1-\sin^2\theta}$$
$$\therefore \frac{1}{\tan^2\theta} = \cot^2\theta = \frac{1-\sin^2\theta}{\sin^2\theta}$$
$$\therefore 1+\cot^2\theta = \frac{\sin^2\theta+1-\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} = \csc^2\theta$$

Chapter 3.3 – Compound angle identities. Double angle identities.

Compound angle identities

Equation booklet:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

Proof

Consider $P(\cos A,\ \sin A)$ and $Q(\cos B,\ \sin B)$ as any two points on the unit circle.
Angle POQ is $A-B$
Using the distance formula:

$$\text{PQ}=\sqrt{(\cos A-\cos B)^2+(\sin A-\sin B)^2}$$

$$\therefore (\text{PQ})^2=\cos^2 A-2\cos A\cos B+\cos^2 B+\sin^2 A-2\sin A\sin B+\sin^2 B$$

$$=\cos^2 A+\sin^2 A+\cos^2 B+\sin^2 B-2(\cos A\cos B+\sin A\sin B)$$

$$=1+1-2(\cos A\cos B+\sin A\sin B)$$

$$=2-2(\cos A\cos B+\sin A\sin B)$$

But according to the cosine rule,

$$(\text{PQ})^2=1^2+1^2-2(1)(1)\cos(A-B)$$

$$=2-2\cos(A-B)$$

$$\therefore \cos(A-B)=\cos A\cos B+\sin A\sin B$$

If we state that $\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\cos\phi$, we can let $-\phi=B$ and $\theta=A$.

$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\cos\phi$ can also be written as:

$$\cos(\theta+(-\phi))=\cos\theta\cos(-(-\phi))+\sin\theta\sin(-(-\phi))$$

$$\therefore \cos(A+B)=\cos A\cos(-B)+\sin A\sin(-B)$$

$$\Rightarrow \sin(-B)=-\sin(B)$$

$$\Rightarrow \cos(-B)=\cos(B)$$

$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\sin(A-B)$$

$$=\cos\left(\frac{\pi}{2}-(A-B)\right)$$

$$=\cos\left(\left(\frac{\pi}{2}-A\right)+B\right)$$

$$=\cos\left(\frac{\pi}{2}-A\right)\cos B-\sin\left(\frac{\pi}{2}-A\right)\sin B$$

$$=\sin A\cos B-\cos A\sin B$$

It is also clear that

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

Double angle identities

$$\sin 2A = \sin(A+A) = \sin A\cos A + \cos A\sin A = 2\sin A\cos A$$

$$\sin 2A = \sin(A+A) = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A$$

$$\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A + \left(\cos^2 A - 1\right) = 2\cos^2 A - 1$$

$$\cos 2A = \cos^2 A - \sin^2 A = \left(1-\sin^2 A\right) - \sin^2 A = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{\tan A + \tan A}{1-\tan A\tan A} = \frac{2\tan A}{1-\tan^2 A}$$

Trig identities table:

Angle	Sine	Cosine	Tangent
0	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{4}}{2}$	0
$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{\sqrt{1}}$
$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{4}}{\sqrt{0}}$

Chapter 3.4 – The circular functions $\sin x$, $\cos x$ and $\tan x$; their domains and ranges; their periodic nature; their graphs. Composite functions of the form $f(x) = a\sin\left(b(x+c)\right)+d$. The inverse functions $x \mapsto \arcsin x$, $x \mapsto \arccos x$, $x \mapsto \arctan x$; their domains and ranges; their graphs.

The period of the untranslated sine and cosine functions is 2π and their range is 1.

In the form $f(x) = a\sin\left(b(x+c)\right)+d$,
a represents a stretch from the *x*-axis of factor *a*,
b represents a stretch from the *y*-axis of factor $\frac{1}{b}$,
c represents a translation *c* units to the left
d represents a translation *d* units upward.

The reciprocal functions can be derived through common sense.

The inverse function can only be derived by restricting the domain to π for all the trigonometric functions, then reflecting in the line $y = x$.

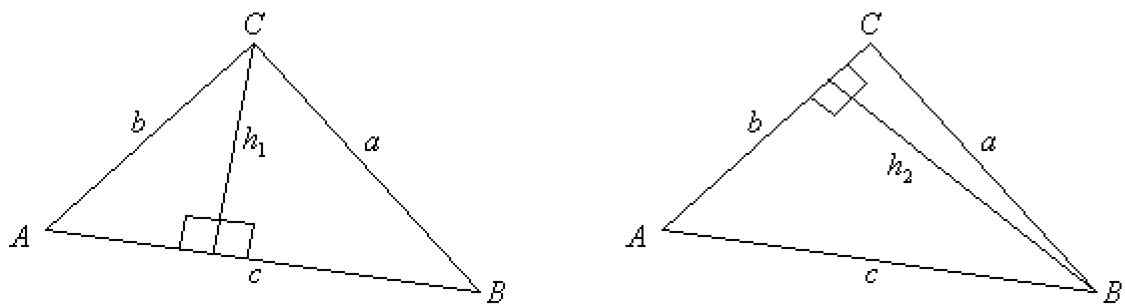
Chapter 3.5 – Solutions of trigonometric equations in a finite interval. Use of trigonometric identities and factorisation to transform equations.

For the first one: Use the inverse functions, then add or subtract 2π (or π) accordingly.

For the second one: Use trigonometric identities to find alternative ways of writing certain transformations. In short, put trigonometric functions into the form $f(x) = a\sin(b(x+c)) + d$ using factors and identities to be able to describe the transformation.

Chapter 3.6 – Solution of triangles. The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$. The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. Area of a triangle as $\frac{1}{2}ab\sin C$.

Any triangle with vertices *A*, *B*, *C* and sides *a*, *b*, *c* can be made into two right-angled triangles as shown.

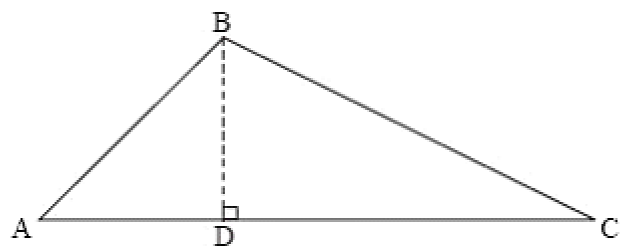


$$\sin A = \frac{h_1}{b}, \quad \sin B = \frac{h_1}{a}$$
$$\therefore b\sin A = h_1 = a\sin B$$
$$\therefore b\sin A = a\sin B$$
$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\sin A = \frac{h_2}{c}, \quad \sin C = \frac{h_2}{a}$$
$$\therefore c\sin A = h_2 = a\sin C$$
$$\therefore c\sin A = a\sin C$$
$$\therefore \frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule can be proved as follows using the triangle below.



$\text{CD} = \text{AC} - \text{AD} = b - c\cos A$

$$\mathrm{BC}^2 = \mathrm{BD}^2 + \mathrm{CD}^2$$
$$\Rightarrow a^2 = (c\sin A)^2 + (b - c\cos A)^2$$
$$= c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A$$
$$= b^2 + c^2\left(\sin^2 A + \cos^2 A\right) - 2bc\cos A = b^2 + c^2(1) - 2bc\cos A$$
$$= b^2 + c^2 - 2bc\cos A$$

Chapter 4.1 – Definition of a matrix: the terms element, row, column and order.

A matrix is a rectangular array of numbers arranged in rows and columns.
A row is a horizontal set of numbers.
A column is vertical set of numbers.
The order of a matrix denotes the number of rows and number of columns in the matrix and is equal to the number of elements in the matrix: $m \times n$ where m is the number of rows and n is the number of columns.

Chapter 4.2 – Algebra of matrices: equality; addition; subtraction; multiplication by a scalar. Multiplication of matrices. Identity and zero matrices.

Two matrices are equal if they are of the same order.
It is only possible to add or subtract two matrices if they are of the same order. Each element of a particular row and column is added or subtracted by the corresponding element in the other matrix.

Multiplication by a scalar involves the mere multiplication of every term in the matrix by that scalar.

Two matrices can only be multiplied together if the first (matrix multiplication is not commutative) matrix has the same number of columns as the second has rows. The resulting matrix has the same number of rows of the first matrix and the same number of columns as the second.

$$\begin{pmatrix} a & b & c \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = ap + bq + cr.$$

The identity matrix I is the matrix such that A×I = A where A is any square matrix. For 2×2 matrices, $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and for 3×3 matrices, $\mathrm{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

To prove this,
Let $\mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

AI = A (by definition). Let $I = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$

Thus:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\therefore \begin{pmatrix} (ap+br) & (aq+bs) \\ (cp+dr) & (cq+ds) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\therefore$

$$ap + br = a$$

$$aq + bs = b$$

$$cp + dr = c$$

$$cq + ds = d$$

$$ap + br = a$$

$$\therefore ap = a - br$$

$$\therefore a = \frac{a - br}{p}$$

$$\therefore ap + br = \frac{a - br}{p}$$

$$\therefore ap^2 + brp = a - br$$

$$\therefore ap^2 - a + brp + br = 0$$

$$\therefore a\left(p^2 - 1^2\right) + br\left(p+1\right) = 0$$

$$\therefore a\left(p+1\right)\left(p-1\right) + br\left(p+1\right) = 0$$

$$\therefore \left(p+1\right)\left(a\left(p-1\right) + br\right) = 0$$

$$\therefore p = 1$$

$$\therefore s = 1$$

$$\therefore r = 0$$

$$\therefore q = 0$$

$$\therefore \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A zero matrix is a matrix in which all elements are equal to 0 such that
AO = **O** and **A** + **O** = **A**

It is important to note that multiplication by both the identity and the zero matrix is commutative.

Here are some rules:

If **A** and **B** are matrices that can be multiplied then **AB** is also a matrix.
Matrix multiplication is non-commutative.
If **O** is a zero matrix then **AO** = **OA** = **O** for all **A**.
A(B + C) = **AB** + **AC**
If **I** is the identity matrix then **AI = IA = A**
$\mathbf{A}^n$ for $n \geq 2$ can be determined provided that **A** is a square and n is an integer.

Chapter 4.3 – Determinant of a square matrix. Calculation of 2×2 and 3×3 determinants. Inverse of a matrix: conditions for its existence.

The inverse of a matrix $\mathbf{A}^{-1}$ is such as will satisfy the equation $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore$$

$$ap+br=1$$
$$aq+bs=0$$
$$cp+dr=0$$
$$cq+ds=1$$

$$ap+br=1$$
$$cp+dr=0$$
$$\therefore$$
$$ap+br=1$$
$$ap+\frac{ad}{c}r=0$$
$$\therefore \left(b-\frac{ad}{c}\right)r=1$$
$$\therefore r=\frac{c}{bc-ad}=\frac{-c}{ad-bc}$$

$$ap=1-br=1-\frac{bc}{bc-ad}=-\frac{ad}{bc-ad}=\frac{ad}{ad-bc}$$
$$\therefore p=\frac{ad}{(ad-bc)a}=\frac{d}{ad-bc}$$
$$\therefore s=\frac{a}{ad-bc}$$
$$\therefore q=-\frac{bc}{(ad-bc)c}=\frac{-b}{ad-bc}$$

$$\therefore \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This suggests that there is no inverse for any matrix where $ad-bc=0$. A matrix without an inverse is known as a singular matrix, and $ad-bc$ is known as the determinant because it determines whether the matrix will be singular or invertible.

The determinant of matrix A is written as $|A|$ and as $\det A$.

Rules: det**AB** = det**A**det**B**.

The determinant of a 3×3 matrix:

Where **A** = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$,

$$|A| = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Chapter 4.4 – Solutions of systems of linear equations (a maximum of three equations in three unknowns). Conditions for the existence of a unique solution, no solution and an infinity of solutions.

$$AX = B$$
$$\therefore X = A^{-1}B$$

Above is how to solve simultaneous equations.

Row Operations (simultaneous equations):

The equations can be interchanged without affecting the solutions
An equation can be replaced by a non-zero multiple of itself
Any equation can be replaced by a multiple of itself ± a multiple of another equation.

Augmented Matrix Form

$\begin{matrix} ax+by=c \\ px+qy=r \end{matrix}$ can be written as $\left[\begin{array}{cc|c} a & b & c \\ p & q & r \end{array}\right]$`

We can now manipulate the equations using row operations to get a matrix in the form $\left[\begin{array}{cc|c} a & b & c \\ 0 & p & q \end{array}\right]$ From which we may get a unique solution.

If a matrix $\left[\begin{array}{cc|c} a & b & c \\ a & b & d \end{array}\right]$ is obtained, the equation has no solution.

If a matrix $\begin{bmatrix} a & b| & c \\ 2a & 2b| & 2c \end{bmatrix}$ is obtained then there are infinitely many solutions.

Reduced row echelon form allows us to find unique solutions to simultaneous equations:

$$\begin{bmatrix} a_1 & b_1 & c_1| & d_1 \\ a_2 & b_2 & c_2| & d_2 \\ a_3 & b_3 & c_3| & d_3 \end{bmatrix} \to \begin{bmatrix} a & b & c| & d \\ 0 & e & f| & g \\ 0 & 0 & h| & i \end{bmatrix}$$

If $h \neq 0$ we arrive at a unique solution.
If $h = 0$ and $i \neq 0$, there is no solution and the system is inconsistent.
If $h = 0$ and $i = 0$, there are infinitely many solutions of the form $x = p + kt$, $y = q + lt$ and $z = t$, $t \in \mathbb{R}$

An underspecified system (not enough equations) is the same case as the last one above. The system may have no solutions, as may be seen by looking for inconsistencies.

If a simultaneous equation in augmented matrix form is invertible, it has a unique solution. If it is singular, it has either no solutions or infinitely many. Converting the augmented matrix to reduced row echelon form allows us to determine which.

Chapter 5.1 – Vectors as displacements in the plane and in three dimensions. Components of a vector; column representation $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$. Algebraic and geometric approaches to the following topics: the sum and difference of two vectors; the zero vector, the vector $-\vec{v}$; multiplication by a scalar, $k\vec{v}$; magnitude of a vector, $|\vec{v}|$; unit vectors $\vec{i}, \vec{j}, \vec{k}$; position vectors $\overline{\mathbf{OA}} = \vec{a}$.

A vector $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ represents a translation of v_1 units along the *x*-axis, v_2 units along the *y*-axis and v_3 units along the *z*-axis. This is because $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors (vectors with a magnitude of 1) where $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and thus represent translations in each of the three axes.

The distance between any two points in three (or two) dimensions is given (by definition) by the magnitude of the vector that maps one of the points onto the other. This is given in the equation $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
For two general points $\mathrm{A}(x_1, y_1, z_1)$ and $\mathrm{B}(x_2, y_2, z_2)$ in a three dimensional space, $|\mathrm{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

The zero vector is a vector $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ such that $0+\vec{a}=\vec{a}=\vec{a}+0$.

If $\vec{v}$ maps point A onto point B, then $-\vec{v}$ maps point B onto point A.

Summary of vector arithmetic

if $\vec{a}=\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\vec{b}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$,

$\vec{a}+\vec{b}=\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix}$

$$\vec{a}+\vec{b}=\vec{b}+\vec{a}$$
$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$
$$\vec{a}+0=0+\vec{a}=\vec{a}$$
$$\vec{a}+-\vec{a}=(-\vec{a})+\vec{a}=0$$

Also, $\left|k\vec{a}\right|=|k||a|$

$$\vec{x}+\vec{a}=\vec{b}\Rightarrow\vec{x}=\vec{b}-\vec{a}$$
$$k\vec{x}=\vec{a}\Rightarrow x=\frac{1}{k}\vec{a}$$

if $\overrightarrow{\mathrm{OA}}=\vec{a}$ and $\overrightarrow{\mathrm{OB}}=\vec{b}$, then $\overrightarrow{\mathrm{AB}}=\vec{b}-\vec{a}$, $\overrightarrow{\mathrm{BA}}=\vec{a}-\vec{b}$
($\overrightarrow{\mathrm{OA}}$ is a position vector, mapping the origin onto a point A).

Definitions

Two vectors are equal if they have the same magnitude and direction, but do not have to be on the same line:
$\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} r \\ s \end{pmatrix}\Leftrightarrow p=r$ and $q=s$

Points are collinear if they lie on the same line:
A, B and C are collinear if $\overrightarrow{\mathrm{AB}}=k\overrightarrow{\mathrm{BC}}$ for some scalar k.

$\vec{a}$ is parallel to $\vec{b}\Leftrightarrow\vec{a}=k\vec{b}$ for some scalar k.

$\frac{1}{\left|\vec{v}\right|}\vec{v}$ is the length of the unit vector in the direction of $\vec{v}$.

Chapter 5.2 – The scalar product of two vectors, $\vec{v} \bullet \vec{w} = |\vec{v}||\vec{w}|\cos\theta$; $\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$. Algebraic properties of the scalar product. Perpendicular vectors. The angle between two vectors.

The scalar product of two vectors, also known as the dot product or inner product of two vectors, $\vec{v} \bullet \vec{w}$ gives us a scalar answer. The scalar product is defined by the second equation, $\vec{v} \bullet \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, but it can be proved that the first equation is also true, $\vec{v} \bullet \vec{w} = |\vec{v}||\vec{w}|\cos\theta$ using the method described on p.381, H&H.

A consequence of $\vec{v} \bullet \vec{w} = |\vec{v}||\vec{w}|\cos\theta$ is that for parallel vectors, where $\theta = 0$, the equation gives $\vec{v} \bullet \vec{w} = |\vec{v}||\vec{w}|\cos 0 = |\vec{v}||\vec{w}|$ and for perpendicular vectors, where $\theta = \frac{\pi}{2}$, the equation gives $\vec{v} \bullet \vec{w} = |\vec{v}||\vec{w}|\cos\frac{\pi}{2} = 0$. Thus, it is possible to find the solution for missing variables by setting $v_1 w_1 + v_2 w_2 + v_3 w_3 = \pm\left(\sqrt{v_1^2 + v_2^2 + v_3^2} \times \sqrt{w_1^2 + w_2^2 + w_3^2}\right)$ and by setting $v_1 w_1 + v_2 w_2 + v_3 w_3 = 0$.

Algebraic properties of the scalar product

$$\vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a}$$

$$\vec{a} \bullet \vec{a} = |\vec{a}|^2$$

$$\vec{a} \bullet \left(\vec{b} + \vec{c}\right) = \vec{a} \bullet \vec{b} + \vec{a} \bullet \vec{c}$$

$$\left(\vec{a} + \vec{b}\right) \bullet \left(\vec{c} + \vec{d}\right) = \vec{a} \bullet \vec{c} + \vec{a} \bullet \vec{d} + \vec{b} \bullet \vec{c} + \vec{b} \bullet \vec{d}$$

Chapter 5.3 – Vector equation of a line $\vec{r} = \vec{a} + \lambda\vec{b}$. The angle between two lines.

In 3-D:

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is the vector equation of a lie where $R(x, y, z)$ is any point on the line and $A(x_0, y_0, z_0)$ is any point on the line. $\vec{b} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is the direction vector of the line (see collinear points). Thus, $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ maps one point in three-dimensional space from the origin and $\lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$ tells us the general position of all collinear points.

The parametric equations of the line, describing the line as a two-dimensional line on the .x, y and z planes respectively, are used when writing the lines in parametric form:

$$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n,\ \lambda \in \mathbb{R}\ .$$

Cartesian form, setting each equation equal to λ and thus equal to one another, gives us the following:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}\ .$$

The angle between the two lines in three dimensional space can be found using the scalar product of their direction vectors:

$$\theta = \arccos \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \times \sqrt{l_2^2 + m_2^2 + n_2^2}}$$ (values taken from the equations of the lines).

Chapter 5.4 – Coincident, parallel and skew lines, distinguishing between these cases. Points of intersection.

Two lines are coincident if the Cartesian equations of one are a scalar multiple of the other.

Two lines are parallel if the angle between the two lines found using the scalar product of their direction vectors is 0 angular units, but the Cartesian equations of one is not a scalar multiple of the others.

Two lines are intersecting if the angle between the two lines found using the scalar product of their direction vectors is θ angular units and they intersect at a point found by representing their equations as matrices and solving them (see chapter 4.4).

Two lines are skew if the angle between the two lines found using the scalar product of their direction vectors is θ angular units and representing their equations as matrices gives no solution (see chapter 4.4).

Another way of finding points of intersection is as follows.

Take two lines L_1 and L_2 where:

$$L_1 :\ x = x_0 + \lambda l_1,\ y = y_0 + \lambda m_1,\ z = z_0 + \lambda n_1$$
$$L_2 :\ x = a_1 + \lambda l_2,\ y = a_2 + \lambda m_2,\ z = a_3 + \lambda n_2$$

If: $\frac{a_1 - x_0}{l_1 - l_2} = \frac{a_2 - y_0}{m_1 - m_2} = \frac{a_3 - z_0}{n_1 - n_2} = \lambda$, the lines intersect at coordinates found by substituting the obtained value of λ into the parametric equations.

Chapter 5.5 – The vector product of two vectors, $\vec{v}\times\vec{w}$. The determinant representation. Geometric interpretation of $\left|\vec{v}\times\vec{w}\right|$.

The vector, or cross product of two vectors, $\vec{v}\times\vec{w}$ is a function of the two vectors which gives a vector perpendicular to the two vectors. Thus, the vector product of two vectors $\vec{v}$ and $\vec{w}$ where $\vec{v}=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\vec{w}=\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ is given by:

$$\vec{v}\times\vec{w}=\begin{pmatrix} v_2w_3-a_3b_2 \\ v_3w_1-v_1w_3 \\ v_1w_2-v_2w_1 \end{pmatrix}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}=\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i}+\begin{vmatrix} v_3 & v_1 \\ w_3 & w_1 \end{vmatrix}\vec{j}+\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}.$$

$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ is known as a 3×3 determinant.

A normal is a line perpendicular to a plane. Thus, given two vectors (or three points) on a plane, a normal to the plane can be found. Since it's a direction vector, any scalar multiple of a normal vector in its simplest form is usable.

Vector product algebra

$$\vec{v}\times\vec{w}\perp\vec{v},\vec{w}$$

$$\vec{v}\times\vec{v}=0$$

$$\vec{v}\times\vec{w}=-\vec{w}\times\vec{v}$$

$\vec{a}\bullet\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and is called the scalar triple product.

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)$$

$$\therefore\left(\vec{a}+\vec{b}\right)\times\left(\vec{c}+\vec{d}\right)=\left(\vec{a}\times\vec{c}\right)+\left(\vec{a}\times\vec{d}\right)+\left(\vec{b}\times\vec{c}\right)+\left(\vec{b}\times\vec{d}\right)$$

$$\left|\vec{v}\times\vec{w}\right|=\sqrt{\left(v_2w_3-v_3w_2\right)^2+\left(v_3w_1-v_1w_3\right)^2+\left(v_1w_2-v_2w_1\right)^2}=\left|\vec{v}\right|\left|\vec{w}\right|\sin\theta,\ \theta=\vec{v}\angle\vec{w}.$$

Properties

If $\vec{u}=\frac{1}{\left|\vec{v}\times\vec{w}\right|}\left(\vec{v}\times\vec{w}\right)$, $\vec{v}\times\vec{w}=\left|\vec{v}\right|\left|\vec{w}\right|\sin\theta\vec{u}$

$\vec{v}\times\vec{w}=0\Leftrightarrow\vec{v}\,||\,\vec{w}.$

If a triangle has defining vectors $\vec{v}$ and $\vec{w}$ then its area is $\frac{1}{2}\left|\vec{v}\times\vec{w}\right|$.

Thus, if a parallelogram has defining vectors $\vec{v}$ and $\vec{w}$ then its area is $\left|\vec{v}\times\vec{w}\right|$.

Chapter 5.6 – Vector equation of a plane $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$. Use of normal vector to obtain the form $\vec{r}\bullet\vec{n}=\vec{a}\bullet\vec{n}$. Cartesian equation of a plane $ax+by+cz=d$.

Since a plane in three dimensional space can be described using a minimum of two lines on the plane, and the vector equation of one line in space is given by $\vec{r}=\vec{a}+\lambda\vec{b}$ where $\vec{a}$ is a vector mapping the origin onto one point on the line and $\vec{b}$ is a the direction vector of the line, the vector equation of the plane can be found by adding the direction vector of another line to the equation, i.e. $\vec{r}=\vec{a}+\lambda\vec{b}+\mu\vec{c}$ where $\vec{b}$ and $\vec{c}$ are two non-parallel vectors that are parallel to the plane.

If A is a point on a plane and R is another point on the plane, then $\overrightarrow{\mathrm{AR}}=\overrightarrow{\mathrm{OR}}-\overrightarrow{\mathrm{OB}}=\vec{r}-\vec{a}$ where $\vec{r}$ is the position vector of R (which is a general point (x,y,z) on the plane) and $\vec{a}$ is the position vector of A. Thus, the normal to the plane will also be perpendicular to that line, so:

$$\vec{n}\bullet\left(\vec{r}-\vec{a}\right)=0$$
$$\therefore\vec{n}\bullet\vec{r}+\vec{n}\bullet\left(-\vec{a}\right)=0$$
$$\therefore\vec{n}\bullet\vec{r}=-\left(\vec{n}\bullet\left(-\vec{a}\right)\right)$$
$$\therefore\vec{n}\bullet\vec{r}=\vec{n}\bullet\vec{a}$$

This is a different way of expressing the vector equation of a line. This means that if a normal vector $\begin{pmatrix}a\\b\\c\end{pmatrix}$ passes through a point (x_0,y_0,z_0) then:

$ax+by+cz=d$, where d is some constant. This gives us the Cartesian equation of the line:

$ax_0+by_0+cz_0=ax+by+cz=d$ where $\begin{pmatrix}a\\b\\c\end{pmatrix}$ is a normal vector of the plane.

Chapter 5.7 – Intersections of: a line with a plane; two planes; three planes. Angle between: a line and a plane; two planes.

When given the Cartesian equation of a plane, the intersection of a line with the plane can be found by using the parametric form for expressing the line in terms x, y, z and λ and substituting each into the Cartesian equation of the plane, thus solving for λ (as shown in H&H, p. 446 example 20).

Take two planes P_1 and P_2 where:

$$P_1:\ x=x_0+\lambda l_1+\mu d_1,\ y=y_0+\lambda m_1+\mu e_1,\ z=z_0+\lambda n_1+\mu f_1$$
$$P_2:\ x=a_1+\lambda l_2+\mu d_2,\ y=a_2+\lambda m_2+\mu e_2,\ z=a_3+\lambda n_2+\mu f_2$$

Thus, the line of intersection of a plane can be found by setting the equations of either *x*, *y* or *z* equal to one another and solving for λ in terms of μ or μ in terms of λ and thus substituting it into the two remaining equations to solve for the remaining constant. The two values found, substituting them into the parametric form equations gives the coordinates of a point of intersection. This method is extremely long and inefficient, however. Keeping the equations in terms of λ will yield a line described in parametric form, however.

The intersection of three planes is found by using the Cartesian forms of the three vectors and inputting them into an augmented matrix to solve for *x*, *y* and *z*. If no solutions are yielded, there is no common point of intersection. If a unique solution is yielded, the three planes meet at a point. If there is are an infinite number of solutions, the planes meet at a line with parametric equations given by the matrix where *z* is substituted by a variable *t*.

Alternatively, if the three planes meet at a point, the inverse matrix method mentioned earlier can be used:

$$\mathrm{AX} = \mathrm{B}$$

$$\therefore \mathrm{X} = \mathrm{A}^{-1}\mathrm{B}$$

Finding whether or not the determinant of A exists can be a quick method for determining whether or not the three planes meet at a point.

The angle between a line with direction vector $\vec{d}$ and a plane with normal vector $\vec{n}$ is found using the equation $\phi = \arcsin\left(\frac{|\vec{n} \bullet \vec{d}|}{|\vec{n}||\vec{d}|}\right)$ (see H&H p.449).

If two planes have normal vectors $\vec{n_1}$ and $\vec{n_2}$, the acute angle θ between the two intersecting planes is given by $\theta = \arccos\left(\frac{|\vec{n_1} \bullet \vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}\right)$. The obtuse angle $\phi = 180 - \theta$.

Chapter 6.1 – Concepts of population, sample, random sample and frequency distribution of discrete and continuous data.

A population is the set of all individuals with a given value for a variable associated with them.

A sample is a small group of individuals randomly selected (in the case of a random sample) from the population as a whole, used as a representation of the population as a whole.

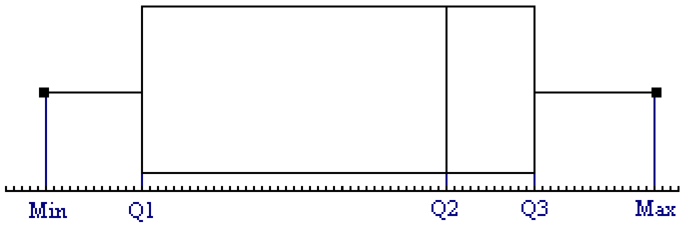
The frequency distribution of data is the number of individuals within a sample or population for each value of the associated variable in discrete data, or for each range of values for the associated variable in continuous data.

Chapter 6.2 – Presentation of data: frequency tables and diagrams, box and whisker plots. Grouped data: mid-interval values, interval width, upper and lower interval boundaries, frequency histograms.

Mid interval values are found by halving the difference between the upper and lower interval boundaries. The interval width is simply the distance between the upper and lower interval boundaries. Frequency histograms are drawn with interval width proportional to bar width and frequency as the height.

Box and whisker plots

A box-and-whisker plot is a visual display of some of the descriptive statistics of a data set. It show The minimum value (Min_x), the lower quartile (Q_1), the median (Q_2), the upper quartile (Q_3) and the maximum value (Max_x). These quantities are known as the five-number summary of a data set.



Chapter 6.3 – Mean, median, mode; quartiles, percentiles. Range; interquartile range; variance, standard deviation.

Mean: $\bar{x}_n = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$

Median: $\tilde{x} = x_m$ where $\frac{\sum_{i=1}^{n} f_i}{2} = f_m$

Mode: $\text{Mode} = x_r$ where $f_r > f_i, i \neq r$

The population mean, μ, is generally unknown but the sample mean, $\bar{x}$ serves as an unbiased estimate of this mean.

A quartile (Q_s) is a the value of x_i which has $\frac{s}{4}$ of the total frequency falling below this value and $\frac{1-s}{4}$ of the total frequency falling above this value. This is only applicable to continuous data.

A percentile is like a quartile, but for $\frac{s}{100}$.

The range is the difference between the highest and lowest value in the data set.

The interquartile range is $Q_3 - Q_1$.

The variance is a measure of statistical dispersion (to what extent the data values deviate from the mean).

The population variance σ^2 of a finite population of size *n* is given by:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

The population variance is, however, generally unknown and hence the adjusted sample variance is used as an unbiased estimate of the population variance:

$s_{n-1}^2 = \frac{n}{n-1}s_n^2$ where s_n^2 is the unadjusted sample variance and s_{n-1}^2 is the estimate.

Chapter 6.4 – Cumulative frequency; cumulative frequency graphs; use to find median, quartiles and percentiles.

Cumulative frequency is the frequency of all values less than a given value. A table can be drawn as shown:

Parameters	f	$\sum f$
$0 \le x < l_1$	a_1	a_1
$l_1 \le x < l_2$	a_2	$a_1 + a_2$
$l_2 \le x < l_3$	a_3	$a_1 + a_2 + a_3$
$l_3 \le x < l_4$	a_4	$a_1 + a_2 + a_3 + a_4$
$l_4 \le x < l_5$	a_5	$a_1 + a_2 + a_3 + a_4 + a_5$
$l_5 \le x < l_6$	a_6	$a_1 + a_2 + a_3 + a_4 + a_5 + a_6$

Drawing a cumulative frequency graph (based on the upper limit of each parameter) enables one to find the median, quartiles and percentiles by taking the required fraction of the total frequency (cumulative frequency of the highest value) and finding the corresponding value on the *x*-axis.

Chapter 6.5 – Concepts of trial, outcome, equally likely outcomes, sample space (U) and event. The probability of an event A as $\mathrm{P}(A) = \frac{n(A)}{n(U)}$. The complementary events A and A' (not A); $\mathrm{P}(A) + \mathrm{P}(A') = 1$.

The number of trials is the total number of times the "experiment" is repeated.

The outcomes are the different results possible for one trial of the experiment.

Equally likely outcomes are expected to have equal frequencies.

The sample space is the set of all possible outcomes of an experiment.

And event is the occurrence of one particular outcome.

$\mathrm{P}(A)=\dfrac{n(A)}{n(U)}$ where $\mathrm{P}(A)$ is the probability of an event A from occurring in one trial, $n(A)$ is the number of members of the event A and $n(U)$ is the total number of possible outcomes.

Since an event must either occur or not occur, the probability of the event *either* occurring *or* not occurring must be 1. This can be stated as follows.

$$\mathrm{P}(A)+\mathrm{P}(A')=1$$

Chapter 6.6 – Combined events, the formula: $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$. $\mathrm{P}(A\cap B)=0$ for mutually exclusive events.

Given two events, *A* and *B*, the probability of *at least* one of the two events occurring, (can also be stated as the probability of either *A or B* occurring) or $\mathrm{P}\left(A\cup B\right)$ is given by the equation $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ where $\mathrm{P}(A)$ is the probability event A occurring, $\mathrm{P}(B)$ is the probability of event *B* occurring and $\mathrm{P}(A\cap B)$ is the probability of both events occurring. It is important to recall (from the product principle) that $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$, where A and B are independent events, or in general $\mathrm{P}(A\cap B)=\mathrm{P}(B)\times\mathrm{P}(A|B)$. This implies that $\mathrm{P}(A\cap B)=0$ for mutually exclusive events *A* and *B* since $P(A|B)$ would be 0 by definition.

Chapter 6.7 – Conditional probability; the definition: $\mathrm{P}(A|B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$. Independent events; the definition: $\mathrm{P}(A|B)=\mathrm{P}(A)=\mathrm{P}(A|B')$.

The two definitions above simply require learning. However the following can thus be derived.

$$\mathrm{P}(A|B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$$
$$\Rightarrow \mathrm{P}(A\cap B)=\mathrm{P}(A|B)\mathrm{P}(B)$$
$$\Rightarrow \mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$$

An important theorem is Bayes' Theorem for two events (not necessarily independent).

$$\mathrm{P}(B|A)=\frac{\mathrm{P}(B)\mathrm{P}(A|A)}{\mathrm{P}(B)\mathrm{P}(B|B)+\mathrm{P}(B')\mathrm{P}(A|B')}.$$

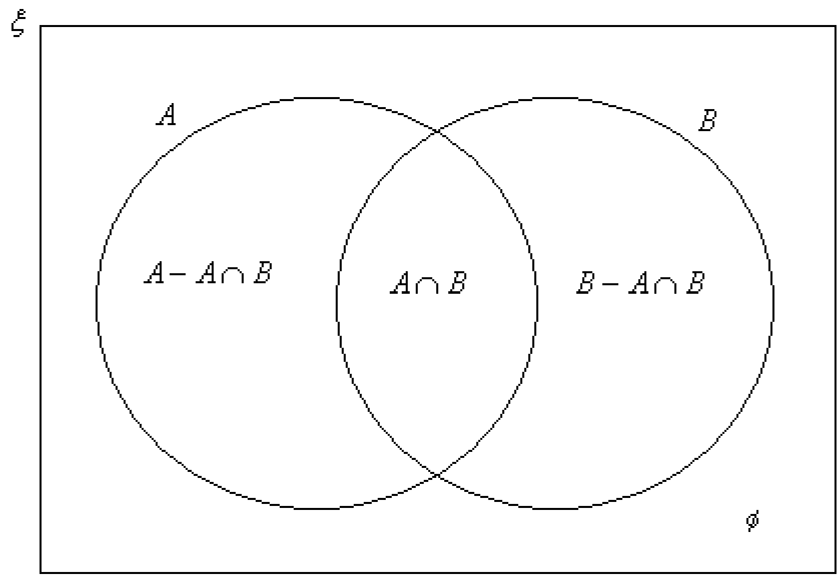
This can be partially derived (or written in another form) in the following way:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{P(A)}$$

Chapter 6.8 – Use of Venn diagrams, tree diagrams and tables of outcomes to solve problems.

Venn Diagrams

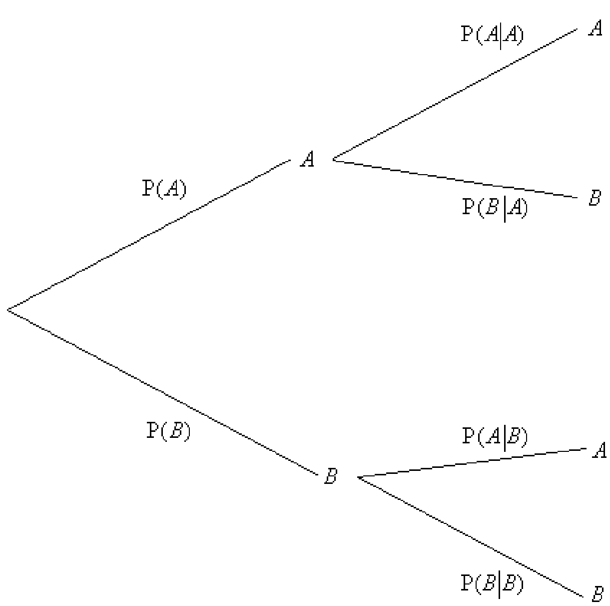


The probability is found using the principle $P(A) = \frac{n(A)}{n(U)}$.

It is important to note that $n(A) = (A - A \cap B) + A \cap B$ and $n(U) = n(A) + n(B) + n(\phi)$.

Tree diagrams

A more flexible method for finding probabilities is known as a tree diagram.



This allows one to calculate the probabilities of the occurrence of events, even where trials are non-identical (where $\mathrm{P}(A|A) \neq \mathrm{P}(A)$), through the product principle.

Tables of outcomes

Refer to H&H p.504-5.

Chapter 6.9 – Concept of discrete and continuous random variables and their probability distributions. Definition and use of probability density functions. Expected value (mean), mode, median, variance and standard deviation.

A random variable represents in number form the possible outcomes, which could occur for some random experiment. For any random variable there is a probability distribution associated with it.

A discrete random variable involves a *count*. Thus, a discrete random variable X has possible values $x_1, x_2, x_3, ...$

Thus, finding $\mathrm{P}(X = x)$ (the probability distribution of x) involves listing $\mathrm{P}(x_i)$ for each value of x_i.

In random distribution, the expected outcome $\mathrm{E}(X)$ is the mean. The standard deviation, or $\mathrm{Var}(X)$, is the square of the distance of X from the mean: $\sigma^2 = \mathrm{Var}(X) = \mathrm{E}(X-\mu)^2 = \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2$. $\mathrm{E}(X^2) = \sum x_i^2 p_i$ for discrete random variables and $\mathrm{E}(X^2) = \int x^2 f(x)\mathrm{d}x$ for continuous random distribution.

For a discrete random variable, the mode and median can be found as outlined above. The mean and variance can be expressed as follows:

$$\mu = \sum x_i p_i$$
$$\sigma^2 = \sum (x_i - \mu)^2 p_i = \sum x_i^2 p_i - \mu^2$$

A continuous random variable involves measurements. A continuous random variable X has all possible values in some interval (on the number line).

Rather than a probability distribution, continuous random variables have probability density functions. A continuous probability function (pdf), $f(x)$, is a function where $f(x) \geq 0$ on a given interval, such as [a, b] and $\int_a^b f(x)\mathrm{d}x = 1$. For a continuous probability density function, the mode is that value of x at the maximum value of $f(x)$ on [a, b]. The median m, is the solution for m of the equation $\int_a^m f(x)\mathrm{d}x = \frac{1}{2}$. The mean and variance can be expressed as follows:

$$\mu = \int \left(x\ f(x)\right)\mathrm{d}x$$
$$\sigma^2 = \int (x-\mu)^2 f(x)\mathrm{d}x = \int x^2 f(x)\mathrm{d}x - \mu^2$$

It is important to note that standard deviation $\sigma = \sqrt{\sigma^2} = \sqrt{\mathrm{Var}(X)}$

The following rules summarise the properties of $\mathrm{E}(X)$.

$\mathrm{E}(k) = k$ for some constant k
$\mathrm{E}(kX) = k\mathrm{E}(X)$ for some constant k
$\mathrm{E}(A(X) + B(X)) = \mathrm{E}(A(X)) + \mathrm{E}(B(X))$ for functions A and B

This makes it possible to derive another form for the variance.

$$Var(X) = \mathrm{E}(X - \mu)^2$$
$$= E(X^2 - 2\mu X + \mu^2)$$
$$= \mathrm{E}(X^2) - 2\mu\mathrm{E}(X) + \mu^2$$
$$= \mathrm{E}(X^2) - 2\mu^2 + \mu^2$$
$$= \mathrm{E}(X^2) - \{\mathrm{E}(X)\}^2$$

Chapter 6.10 – Binomial distribution, its mean and variance. Poisson distribution, its mean and variance.

Binomial Distribution

In the case of n trials where there are r successes and $n - r$ failures, $\mathrm{P}(X = r) = C_r^n p^r q^{n-r}$ where $q = 1 - p$ and $r = 0, 1, 2, 3, 4, \ldots, n$.
p is the probability of a success and q is the probability of a failure. $\mathrm{P}(X = r)$ is the probability distribution function.

There are three criteria that must be met in order for a random probability distribution to be a binomial distribution.

1. The probability distribution is discrete.
2. There are two outcomes – success and failure.
3. The trials are independent – the probability of success is a constant in each trial.

If x is a random variable which is binomial with parameters n and p, then the mean of x is $\mu = np$ and the variance of x is $\sigma^2 = npq$.

Calculator: $\mathrm{P}(x = r) = \mathrm{binompdf}(n, p, r)$ and $\mathrm{P}(x \le r) = \mathrm{binomcdf}(n, p, r)$

Poisson Distribution

The Poisson distribution is defined as $\mathrm{P}(X = x) = \dfrac{m^x e^{-m}}{x!}$ where m is the parameter.

$$m = \frac{\sum f\ x}{\sum f} = \mu = \sigma^2 .$$

There are three criteria that must be met in order for a random probability distribution to be a binomial distribution.

1. The average number of occurrences (μ) is constant for every interval.
2. The probability of more than one occurrence in a given interval is very small.
3. The number of occurrences in disjoint intervals are independent of each other.

Chapter 6.11 – Normal distribution. Properties of the normal distribution. Standardization of normal variables.

If X is normally distributed then its probability density function is given by $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ for $-\infty < x < \infty$.

The grand majority of continuous distributions are normal distributions, where the probability density decreases according to how far the value is from the mean. This is particularly true for variables in nature.

Properties

The curve is symmetrical about the line $x=\mu$

$\lim_{x\to\pm\infty} f(x)=0$

$\int_{-\infty}^{\infty} f(x)\mathrm{d}x=1$

$\mu=\mathrm{Max}\left(f(x)\right)$

σ is uniquely determined as the horizontal distance from the vertical distance from the vertical line $x=\mu$ to a point of inflection.

In a normal distribution, 68.26% of values lie within one standard deviation of the mean, 95.4% of values lie within two standard deviations of the mean and 99.74% of values lie within three standard deviations of the mean.

Since $\mathrm{P}(A)=\frac{n(A)}{n(U)}$, the probability of X lying within a certain interval is equal to the percentage of values that lie within that interval. This is obtained from the data booklet and from GDCs.

The standard normal distribution, or Z-distribution, is the application of the transformation $Z=\frac{X-\mu}{\sigma}$ to a normal X-distribution, such that the mean is at $x=0$ and there is one standard deviation per unit on the x-axis. Where the probability density function for normal distribution has two parameters μ and σ, the Z-distribution has none. This makes it useful when comparing results from two or more different normal distributions, since comparing Z-values allows one to take into account the standard deviation and mean when comparing results.

Finding probabilities with a GDC involves using $\mathrm{normalcdf}(a,b,\mu,\sigma)$ for lower limit a and upper limit b (under "DISTR"). It is important to note that $\mathrm{P}(Z\le a)=\mathrm{P}(Z<a)$.

To find probabilities for a normally distributed random variable X, convert X values to Z using the transformation, sketch the standard normal curve (shade the required region) and find the standard normal table or a graphics calculator to find the probability. $\mathrm{normalpdf}(x,\mu,\sigma)$ gives the probability for a particular x-value.

Chapter 7.1 – Informal ideas of limit and convergence. Definition of derivative as $f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$. Derivative of x^n $(n\in\mathbb{Q})$, $\sin x$, $\cos x$, $\tan x$, e^x and $\ln x$. Derivative interpreted as a gradient function and as rate of change. Derivatives of reciprocal circular functions. Derivatives of a^x and $\log_a x$. Derivatives of $\arcsin x$, $\arccos x$, $\arctan x$.

All functions approach a particular value as the value of the variable they are in terms of approaches a given value. However, in certain cases, it is not possible to directly determine the value of the function at that particular value of the variable because the answer involves division by zero or the value of the variable in question is, in fact, infinity. Thus, where a function converges towards a particular value, it can be said that the limit of the function at that value of the variable is equal to the value the function approaches.

For example, $\sin\theta=0$ only where $\theta=0$ and since the function $\sin\theta$ is continuous, it means that for $\theta\approx 0, \sin\theta\approx\theta$. What is more, since the closest point of inflection to the sine function occurs at 0, the closer θ is to 0, the closer $\sin\theta$ is to θ.

Thus, given the function $f(\theta)=\frac{\sin\theta}{\theta}$, the closer θ is to 0, the closer $f(\theta)=\frac{\theta}{\theta}=1$. However, $\frac{0}{0}$ is undefined. Thus, we can say that since $f(\theta)$ tends towards 1 as θ tends towards 0, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$, or the limit of $\frac{\sin\theta}{\theta}$ as θ tends to 0 is 1.

The derivative, $f'(x)$ of a point $(f(x),x)$ on a function is the gradient of the tangent – or instantaneous rate of change of the function – at that point. This can be found by using the area of limits:

Take two points A and B on the curve of a function $f(x)$. Let A have coordinates $(f(x),x)$ and B, h units away from A on the x-axis, therefore have coordinates $(f(x+h),(x+h))$. Thus, the gradient of the arc AB on the curve joining the two lines is equal to $\frac{f(x+h)-f(x)}{(x+h)-x}=\frac{f(x+h)-f(x)}{h}$. It can be seen that the closer point B is to A, the closer the gradient of the arc is to the gradient of the tangent at point A. Expressed using limit notation, this gives us the equation:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Using this equation to find the derivative of a function at a point is known as finding the derivative from first principles. This is done by manipulating the equation until h is taken out of the denominator, since this will generate exactly the same result as the previous equation but with the additional solution for where $h=0$, thus telling us what the limit of the function as h tends towards 0 is at the point by setting the $h=0$ and solving for the derivative.

This allows us to generate some general rules for the derivatives.

The first of these is the power rule:

Let $f(x)=x^n$
Thus:

$$f'(x)=\lim_{h\to0}\frac{(x+h)^n-x^n}{h}=\lim_{h\to0}\frac{x^n+\binom{n}{1}x^{n-1}h+\ldots+\binom{n}{r}x^{n-r}h^r+\ldots+h^n-x^n}{h}$$

$$\therefore\lim_{h\to0}\frac{\binom{n}{1}x^{n-1}h+\ldots+\binom{n}{r}x^{n-r}h^r+\ldots+h^n}{h}=\lim_{h\to0}\binom{n}{1}x^{n-1}+\ldots+\binom{n}{r}x^{n-r}h^{r-1}+\ldots+h^{n-1}$$

$$=\binom{n}{1}x^{n-1}+0=nx^{n-1}$$

This result is known as the power rule.

Also note:

If $f(x)=ax^n$,

$$f'(x)=\lim_{h\to0}\frac{a(x+h)^n-ax^n}{h}=\lim_{h\to0}\frac{a\left((x+h)^n-x^n\right)}{h}=a\lim_{h\to0}\frac{\left((x+h)^n-x^n\right)}{h}=anx^{n-1}$$

It is also fairly evident that any constant if $f(x)=ax^n+k$, $f'(x)=anx^{n-1}$

The following are rules that are important to note:

$$f(x)=ax^n\Leftrightarrow f'(x)=anx^{n-1}$$
$$f(x)=\sin x\Leftrightarrow f'(x)=\cos x$$
$$f(x)=\cos x\Leftrightarrow f'(x)=-\sin x$$
$$f(x)=\tan x\Leftrightarrow f'(x)=\sec^2 x$$
$$f(x)=e^x\Leftrightarrow f'(x)=e^x$$
$$f(x)=\ln x\Leftrightarrow f'(x)=\frac{1}{x}$$

When expressed as $f'(x)$, the derivative of $f(x)$ suggests the rate of change function. When expressed as $\frac{\mathrm{d}y}{\mathrm{d}x}$, the derivative of $f(x)=y$ suggests a gradient function. The two are, however, completely interchangeable, though $\frac{\mathrm{d}y}{\mathrm{d}x}$ is by far a more useful and easier to manipulate form.

This is because $\frac{\mathrm{d}y}{\mathrm{d}x}$ represents the point gradient on the curve as the ratio between an infinitely small displacement dy in the y-direction and an infinitely small displacement dx in the x-direction, representing any given curve as a series of connected infinitely small line segments with gradient equal to the tangent of the curve at that point.

Because this representation of the gradient of the tangent is in the form of a fraction (of infinitely small parts), this allows it to be manipulated in such a way as to yield interesting results.

For instance, where $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives the gradient function of a curve whose y values are in terms of x, $\frac{1}{\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)$ gives the gradient function of a curve whose x-values are in terms of y. Taking the integral of this function gives you $\int \frac{\mathrm{d}x}{\mathrm{d}y} \bullet \mathrm{d}x$ which is not a very useful result, however the inverse of this function, $\int \frac{\mathrm{d}x}{\mathrm{d}y} \bullet \mathrm{d}y$ gives us (what used to be) x in terms of y, in other words, the inverse of the function.

To summarise: The inverse of the integral of the reciprocal of the derivative of a function is equal to the inverse of the function.
The inverse of the reciprocal of the derivative of a function is equal to the derivative of the inverse of the function.

The above derivatives and the many more required are all in the formula booklet.

Increasing and decreasing functions.

An increasing function has a positive gradient for all x and a decreasing function has a negative gradient for all x. The intervals during which a function is increasing or decreasing is found by finding the gradient function and using sign diagrams to determine when the gradient function is positive and when it is negative.

If a curve has gradient function $\frac{\mathrm{d}y}{\mathrm{d}x}$, the normal to the curve has function $-\frac{\mathrm{d}x}{\mathrm{d}y}$.

Chapter 7.2 – Differentiation of a sum and a real multiple of functions in 7.1. The chain rule for composite functions. Application of chain rule to related rates of change. The product and quotient rules. The second derivative. Awareness of higher derivatives.

An important rule to remember in differentiation is that if:

$y = a \bullet f(x) + b \bullet g(x) + \ldots + c \bullet h(x)$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = a \bullet f'(x) + b \bullet g'(x) + \ldots + c \bullet h'(x)$

The chain rule takes advantage of the fractional properties of the gradient equation to simplify the differentiation of functions such as $y = \left(f(x)\right)^n$ and to allow for the determination of other related rates of change.

The chain rule is perhaps best described as it is in the formula booklet:

$y = g(u)$, where $u = f(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$

$$y = f(g(x)) \Leftrightarrow \frac{dy}{dx} = g'(x) \bullet f(g(x))$$

$$y = f(u) \Leftrightarrow \frac{dy}{dx} = u' \bullet f(u),\ u = g(x)$$

This has a great many applications, not only to allow for the differentiation of more complicated functions but also to allow for the derivation of many other functions.
The Chain rule also allows for the determination of related rates of change.

$\frac{dv}{dt} = \frac{dv}{ds} \times \frac{ds}{dt}$ is one example of this, where the function for instantaneous acceleration is using the functions of instantaneous velocity and the equation of speed relative to displacement.

The product rule can be described in a similarly clear way:

$$y = uv,\ \text{where } u = f(x), v = g(x) \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

As can the quotient rule:

$$y = \frac{u}{v},\ \text{where } u = f(x), v = g(x) \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Generally speaking, it is safe to say that these are merely equations to be directly applied to the question as needed.

The second derivative $f''(x)$ or $\frac{d^2y}{dx^2}$ is the derivative of the derivative of the function. It represents the curvature of the function: the rate at which the gradient is changing in relation to x. This is useful for reasons outlined in the next chapter.

Higher derivatives, expressed as $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$, are the derivatives of the derivative one order down.

Chapter 7.3 – Local maximum and minimum points. Use of the first and second derivative in optimization problems.

Local maxima and minima occur where $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} \neq 0$. If $\frac{d^2y}{dx^2} > 0$ for that value of x, the point is a minimum, if $\frac{d^2y}{dx^2} < 0$ for that value of x, the point is a maximum. These can therefore be used in optimisation questions where the function for a given parameter is found and the differentiation applied, such as in cases dealing with profit, area or volume.

Chapter 7.4 – Indefinite integration as antidifferentiation. Indefinite integral of x^n $(n \neq -1)$, $\sin x$, $\cos x$, e^x, $\frac{1}{x}$. The composites of any of these with the linear function $ax+b$.

Indefinite integration gives the general formula for the area under a function from the origin to *x*. One form of indefinite integration is antidifferentiation. It can be explained conceptually in the following way. Seeing as differentiation is the process by which you divide the infinitely small rise by the infinitely small run at each point, the reverse process, defined as antidifferentiation, must involve the opposite process, as in finding the sum the areas of the infinitely small trapeziums under each infinitely small line segment.

Antidifferentiation is simply the inverse function of differentiation. The antidifferential of a function is the function which, when differentiated, gives the original function.

When asked to antidifferentiate, ask yourself: What function, when differentiated, would give me this function?

Integration rules:

$$\int kf(x) \bullet dx = k\int f(x) \bullet dx$$

$$\int [f(x)+g(x)] \bullet dx = \int f(x) \bullet dx + \int g(x) \bullet dx$$

$$\int k \bullet dx = kx + c$$

$$\int x^n \bullet dx = \frac{1}{n+1} x^{n+1} + c$$

$$\int e^x \bullet dx = \int e^x + c$$

$$\int \frac{1}{x} \bullet dx = \ln|x| + c$$

$$\int e^{ax+b} \bullet dx = \frac{1}{a} e^{ax+b} + c$$

$$\int (ax+b)^n \bullet dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$$

$$\int \frac{1}{ax+b} \bullet dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int f(u)\frac{du}{dx} dx = \int f(u) \bullet du$$

Chapter 7.5 – Anti-differentiation with a boundary condition to determine the constant term. Definite Integrals. Area between a curve and the *x*-axis or *y*-axis in a given interval, areas between curves. Volumes of revolution.

If given the value of $f(x)$ at a given value of *x* for the integral of a function, it is possible to plug in the numbers to determine the constant term *c* in $\int f'(x)\mathrm{d}x = f(x) + c$.

$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a)$ where b is the upper limit of x and a is the lower limit.

This gives the area between the curve and the x-axis for those limits. To find the area between the curve and the y-axis, it is simplest to take the area of the rectangle $f'(b)\times b$ and subtract from that $\int_a^b f'(x)\mathrm{d}x$ and the area of rectangle $f'(a)\times a$. It is also possible to find the area under the curve of the inverse function, i.e. $\int_a^b x\mathrm{d}y$ rather than $\int_a^b y\mathrm{d}x$.

For the volume V of revolution when an area with limits a and b is rotated about the x- (first case) or y-axis (second case), it is simplest to state the equations:

$$V=\int_a^b \pi y^2\mathrm{d}x,\ V=\int_a^b \pi x^2\mathrm{d}y$$

Similarly to the usage of rectangles outlined above, cylinders can be used when rotating an area between the curve and the axis is not the axis of revolution.

If rotating above a line that is not on an axis, it is necessary to use transformations to transform the axis of rotation onto either the x- or y-axis in order to use the equations shown above.

Chapter 7.6 – Kinematic problems involving displacement, s, velocity, v, and acceleration, a.

$$v=\frac{\mathrm{d}s}{\mathrm{d}t},\ a=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=v\frac{\mathrm{d}v}{\mathrm{d}s}$$

The area under a velocity-time graph represents distance. It can be important to manipulate derivatives so that they are in the correct form to find the required solution. This manipulation must not be forgotten.

Chapter 7.7 – Graphical behaviour of functions: tangents and normals, behaviour for large $|x|$; asymptotes. The significance of the second derivative; distinction between maximum and minimum points. Point of inflexion with zero and non-zero gradients.

The derivative of a function gives the function of its tangent, the negative reciprocal of the derivative gives the function of its normal or the normal at that point. The derivative indicates where the function is getting more positive and where it is getting more negative. Vertical asymptotes occur where the derivative has infinite value and the equation of the normal has value 0 and horizontal asymptotes occur where the derivative of the inverse function is infinite or where the tends to zero as x approaches $\pm\infty$ (and where the function of the normal approaches infinity).

The second derivative gives the rate of change of the derivative, i.e. the curvature of the function (whether the function is getting steeper or less steep). At stationary points, if the second derivative is positive, the stationary point is a minimum, if the second derivative is negative, the stationary point is a maximum and where the stationary point has second derivative 0, the stationary point is a point of inflection, given that the third derivative has a non-zero value.

Chapter 7.8 – Implicit differentiation.

$$\frac{dy}{dx}x^m y^n = mx^{m-1}y^n + x^m ny^{n-1}\frac{dy}{dx}$$

Chapter 7.9 – Further integration: integration by substitution integration by parts.

Integration by substitution involves taking $\int g(f(x))\mathrm{d}x$, replacing $f(x)$ by u and multiplying by $\frac{\mathrm{d}u}{\mathrm{d}x}$. This is derived through reverse application of the chain rule. The part of the function chosen to serve as u depends on the situation and a feel for it can best be obtained through practice. CHAPTER 29 H&H

Integration by parts involves the use of the equation:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Sometimes it may be necessary to do repeated integration by parts to obtain the answer.

Chapter 7.10 – Solutions of first order differential equations by separation of variables.

This is best understood by referring to exercise 29D in H&H. It is best summarised as shown:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky + c$$
$$\Rightarrow \frac{1}{ky} \bullet \frac{\mathrm{d}y}{\mathrm{d}x} = c$$
$$\Rightarrow \frac{1}{ky} \bullet \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = c\mathrm{d}x$$
$$\Rightarrow \int \frac{1}{ky}\mathrm{d}y = \int c\mathrm{d}x$$
$$\Rightarrow \frac{1}{k}\ln y = cx + q$$
$$\Rightarrow \ln y = kcx + kq$$
$$\Rightarrow y = e^{kcx+kq}$$
$$= A^k e^{kcx}$$

This is not a general equation however the general method can be used to solve other types of equation.