Calculus (1): Differentiation

9

Calculus is a very important branch of Mathematics. It was developed by Newton (1642–1727) and Leibnitz (1646–1716) to deal with changing quantities. The gradient of a curve is an example of such a quantity and we begin with this.

GRADIENT OF A CURVE

The gradient of a straight line is constant. It is equal to the ratio $\frac{y\text{-step}}{x\text{-step}}$ between any two points of the line (see Chapter 1, page 8). On a curve however, the gradient is changing from one point to another. We define the gradient at any point on a curve therefore to be the gradient of the **tangent to the curve at that point (Fig.9.1).**

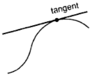

Fig.9.1

We now find a **gradient function,** derived from the function represented by the curve, using a method called a **limiting process.** Consider the simple quadratic curve $y = x^2$ (Fig.9.2) and take the point P(3,9) on that curve.

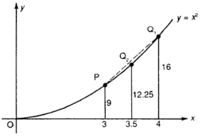

Fig.9.2

Now take a point near P, say $Q_1(4,16)$. The gradient of the line PQ_1 is $\frac{16-9}{1} = 7$ which only approximately equals the gradient of the tangent at P.

To get a better approximation we try again, this time with $Q_2(3.5,12.25)$ which is closer to P.

The gradient of $PQ_2 = \frac{3.25}{0.5} = 6.5$.

Now see what happens if we repeat this, taking positions of Q closer and closer to P, using a calculator (Fig.9.3).

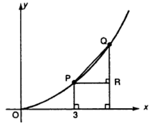

Fig.9.3

Coordinates of Q	*QR*	*PR*	*Gradient*	
(3.3,10.89)	1.89	0.3	6.3	
(3.1,9.61)	0.61	0.1	6.1	
(3.01,9.0601)	0.0601	0.01	6.01	tending to 6
(3.001,9.006001)	0.006001	0.001	6.001	
(3.0001,9.00060001)	0.00060001	0.0001	6.0001	
.	.	.	.	
.	.	.	.	

The sequence of values suggests that as we continue, taking Q closer and closer to P, the gradient approaches 6. We say that 6 is the **limiting value or limit** of the sequence. As Q ⟶ (tends to) P, the gradient of PQ ⟶ 6 and **we take this limiting value as the gradient at P.**

Note that we cannot find this value *directly*. We have to use this limiting method. (We also have to be sure that there will be a limit but this will be assumed in our work).

Exercise 9.1 *(Answers on page 626.)*

1 Repeat the limiting process to find the gradient where $x = 2$ and $x = -1$ on the curve $y = x^2$.

2 Use the limiting method to find the gradient at the point where $x = 2$ on the curve $y = 2x^2$.

3 By the same method, find the gradient where $x = 4$ on the curve $y = \sqrt{x}$.

GENERAL METHOD FOR THE GRADIENT FUNCTION

To find the gradient at another point on the curve we must repeat the calculations. A better approach would be to find a formula for the gradient, using the same method.

In Fig.9.4, we take a general point P whose coordinates are (x,x^2).

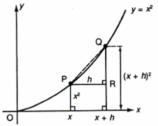

Fig.9.4

Now take a nearby point Q where $x_Q = x + h$. At present the value of h is not specified except that $h \neq 0$.

Then $y_Q = (x + h)^2$.

$$\text{Gradient of PQ} = \frac{QR}{PR} = \frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h \quad (h \neq 0)$$

Now suppose Q moves closer and closer to P, i.e. $h \longrightarrow 0$. Then $2x + h \longrightarrow 2x$. The limiting value of $2x + h$ is $2x$ and we take this as the gradient at P. When $x = 3$, the gradient = 6, as we found before. When $x = 0$, the gradient is 0, which can be seen from the graph as this is the turning point.

The function $x \longmapsto 2x$ is the **gradient function** for the curve $y = x^2$. Each curve will have its own gradient function which we find by the limiting method, known as working from first principles.

Example 1

Find the gradient function for $y = 2x^2 - 3$ (Fig.9.5).

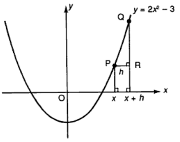

Fig. 9.5

P is a general point $(x, 2x^2 - 3)$.
Q is a nearby point with $x_Q = x + h$, $y_Q = 2(x + h)^2 - 3$.

$$QR = 2(x + h)^2 - 3 - (2x^2 - 3)$$
$$= 2x^2 + 4xh + 2h^2 - 3 - 2x^2 + 3$$
$$= 4xh + 2h^2$$

$$\frac{QR}{PR} = \frac{4xh + 2h^2}{h} = 4x + 2h$$

Now as $h \longrightarrow 0$, $4x + 2h \longrightarrow 4x$. The limiting value is $4x$. The gradient function is therefore $x \longmapsto 4x$.

Example 2

(a) Find the gradient function for the curve $y = x^3 + 2x$ (Fig.9.6).
(b) Hence find the gradients at $x = 0$ and $x = -1$.
(c) Is there a value of x where the gradient is 0?

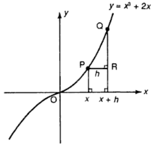

Fig.9.6

(a) Take P as $(x,x^3 + 2x)$ and a nearby point Q where $x_Q = x + h$, $y_Q = (x + h)^3 + 2(x + h)$.

Then $QR = (x + h)^3 + 2(x + h) - (x^3 + 2x)$
$= x^3 + 3x^2h + 3xh^2 + h^3 + 2x + 2h - x^3 - 2x$
$= 3x^2h + 3xh^2 + h^3 + 2h$

Gradient of $PQ = \frac{QR}{PR} = \frac{3x^2h + 3xh^2 + h^3 + 2h}{h} = 3x^2 + 3xh + h^2 + 2$

As $h \longrightarrow 0$, $3xh$ and h^2 each $\longrightarrow 0$. The limiting value is $3x^2 + 2$ and the gradient function is $x \longmapsto 3x^2 + 2$.

(b) When $x = 0$, the gradient = 2; when $x = -1$, the gradient = 5.

(c) The equation $3x^2 + 2 = 0$ has no solution so the gradient of the curve is never 0.

NOTATION

If the equation of the curve is $y = f(x)$, we write the gradient function as $\mathbf{f'(x)}$.

If we take a point P on $y = f(x)$ (Fig.9.7), the coordinates of P are $(x,f(x))$. The coordinates of Q are $(x + h, f(x + h))$.

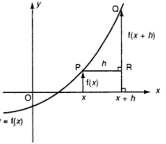

Fig.9.7

Then $QR = f(x + h) - f(x)$ and the gradient of $PQ = \frac{f(x + h) - f(x)}{h}$.

The limiting value of $\frac{f(x + h) - f(x)}{h}$ as $h \longrightarrow 0$ will be $f'(x)$.

We write this as
$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$\lim\limits_{h \to 0}$ means 'take the limiting value when $h \longrightarrow 0$'.

Example 3

If $f(x) = \frac{1}{x-1}$ $(x \neq 1)$*, find* $f'(x)$.

$$f(x+h) - f(x) = \frac{1}{x+h-1} - \frac{1}{x-1}$$

$$= \frac{x-1-(x+h-1)}{(x+h-1)(x-1)}$$

$$= \frac{-h}{(x+h-1)(x-1)}$$

Then $\frac{f(x+h)-f(x)}{h} = \frac{-1}{(x+h-1)(x-1)}$

As $h \longrightarrow 0$, $x + h - 1 \longrightarrow x - 1$.

So the limiting value of $\frac{f(x+h)-f(x)}{h}$ will be $\frac{-1}{(x-1)^2}$ and $f'(x) = \frac{-1}{(x-1)^2}$.

Note that this is always negative, so the gradient on the curve is always negative, as seen from the sketch (Fig.9.8).

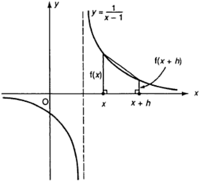

Fig.9.8

Exercise 9.2 *(Answers on page 627.)*

1 Find the gradient function for the following curves:

(a) $y = 3x^2$ (b) $y = 3x^2 - 1$ (c) $y = x^2 + x - 1$

(d) $y = 2 - x^2$ (e) $y = \frac{1}{x}$ $(x \neq 0)$ (f) $y = x^3 - x^2 + 1$

(g) $y = \frac{1}{x-1}$

2 For each of the curves in Question 1, find the gradient where (i) $x = 2$, (ii) $x = -1$, (iii) $x = 0$ (except curve (e)).

3 (a) Find the gradient function for $y = 3x^2 - 6x + 2$.

(b) For what value of x is the gradient 0?

(c) Hence find the minimum value of the function.

4 (a) If $f(x) = x^3 - x^2$, find $f'(x)$.
(b) For what values of x is the gradient on the curve $y = f(x)$ zero?
(c) Find the values of $f(x)$ at these points.

5 By finding the gradient function, show that the curve $y = 1 - 4x - x^2$ has a turning point where $x = -2$. Is this a maximum or minimum point?

6 Find the gradient function for the curve $y = \frac{1}{x} + 4x$ $(x \neq 0)$.
Hence find the values of x where the gradient on this curve is zero.

7 Find the gradient function for $y = ax^2 + bx + c$ where a, b and c are constants.

The δy, δx Notation for the Gradient Function

To find $f'(x)$, we took two points whose x-coordinates were x and $x + h$. We now introduce a new and important notation. Instead of h, we write δx (read *delta x*) which is *one* symbol for the change in x, called the **increment** in x.

We use the curve $y = x^2$ again (Fig.9.9). Now if x changes to $x + \delta x$, y will also change to $y + \delta y$, where δy is the corresponding increment in y.

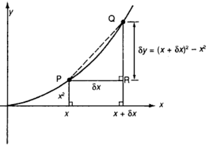

Fig.9.9

$PR = \delta x$, $QR = \delta y$.

The coordinates of Q are $(x + \delta x, y + \delta y)$ and so $y + \delta y = (x + \delta x)^2$.

$$QR = (x + \delta x)^2 - x^2$$
$$= x^2 + 2x\delta x + (\delta x)^2 - x^2$$
$$= 2x\delta x + (\delta x)^2$$

The gradient of PQ $= \dfrac{\delta y}{\delta x} = \dfrac{2x\delta x + (\delta x)^2}{\delta x} = 2x + \delta x$

Now we let $\delta x \longrightarrow 0$. The limiting value of $\frac{\delta y}{\delta x}$ will be $2x$, so the gradient function is $2x$ as before.

The special feature of this notation is that we write the gradient as $\frac{dy}{dx}$ (read *dee y by dee x*) to symbolize the limiting value of $\frac{\delta y}{\delta x}$ as $\delta x \longrightarrow 0$.
(The curly δ is straightened to ordinary d to show that we have taken the limiting value).

So $$f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

This δy, δx notation will be used from now on.

Note: $\frac{dy}{dx}$ is to be taken as one symbol and NOT as a fraction.

Example 4

If $y = 16x + \frac{1}{x^2}$, *find* $\frac{dy}{dx}$ *from first principles.*
Find the value of x where the gradient is 0.

$f(x) = 16x + \frac{1}{x^2}$ and $f(x + \delta x) = 16(x + \delta x) + \frac{1}{(x + \delta x)^2}$

Hence $f(x + \delta x) - f(x) = 16x + 16\delta x + \dfrac{1}{(x + \delta x)^2} - 16x - \dfrac{1}{x^2}$

$$= 16\delta x + \frac{x^2 - (x + \delta x)^2}{(x + \delta x)^2 x^2}$$

$$= 16\delta x + \frac{x^2 - x^2 - 2x\delta x - (\delta x)^2}{(x + \delta x)^2 x^2}$$

$$= 16\delta x - \frac{2x\delta x + (\delta x)^2}{(x + \delta x)^2 x^2}$$

Then $\dfrac{\delta y}{\delta x} = 16 - \dfrac{2x + \delta x}{(x + \delta x)^2 x^2}$

As $\delta x \longrightarrow 0$, $2x + \delta x \longrightarrow 2x$ and $x + \delta x \longrightarrow x$.

Then $\dfrac{dy}{dx} = \lim_{\delta x \to 0} \dfrac{\delta y}{\delta x} = 16 - \dfrac{2x}{x^4} = 16 - \dfrac{2}{x^3}$

When the gradient is 0, $\frac{dy}{dx} = 16 - \frac{2}{x^3} = 0$ i.e. $16x^3 = 2$ or $x^3 = \frac{1}{8}$ giving $x = \frac{1}{2}$.

Exercise 9.3 *(Answers on page 627.)*

1 Find $\frac{dy}{dx}$ from first principles for

(a) $y = 3x^2 + 1$ (b) $y = 2 - 4x^2$ (c) $y = 4x^3$

(d) $y = x - \frac{1}{x}$ (e) $y = \frac{x^2}{2} - x + 3$ (f) $y = \frac{x^3}{3}$

2 What is the gradient of $y = 5$? Hence explain why $\frac{dy}{dx} = 0$ if $y = k$ (a constant).

3 Given $y = 2x^2 - 4x + 1$, find (a) $\frac{dy}{dx}$ and (b) the coordinates of the point on the curve where the gradient is (i) 0, (ii) –8.

4 (a) Find $\frac{dy}{dx}$ if $y = x^2 + ax + 3$ where a is a constant.

(b) Find the value of a if the gradient where $x = 3$ is 2.

MEANING OF $\frac{dy}{dx}$

The notation $\frac{dy}{dx}$ for the limiting value of $\frac{\delta y}{\delta x}$ as $\delta x \longrightarrow 0$ is appropriate as it is a reminder that $\frac{dy}{dx}$ is derived from $\frac{\delta y}{\delta x}$. We call this **differentiation** (as it uses the difference δx) and $\frac{dy}{dx}$ is called the **derivative** or the **differential coefficient** of y with respect to x. We shall use the abbreviation 'wrt' for 'with respect to'.

$\frac{dy}{dx}$ gives the gradient function for a curve and the value of $\frac{dy}{dx}$ at a given point is the gradient of the curve and therefore of the tangent there.

Now the gradient at a point measures the rate at which y is changing wrt x. The steeper the gradient the greater this rate of change. For example, on the curve $y = x^2$, the two quantities are each changing and the rate of change is $2x$. When $x = 3$, $y = 9$ but y is changing at that point 6 times as much as x is changing. $\frac{dy}{dx}$ measures this rate of change. This is what makes differentiation a powerful tool in Mathematics.

The idea and notation can be applied to any function. For example, if s is a function of t, $s = f(t)$, then $\frac{ds}{dt}$ is the rate of change of s wrt t.
If A is a function of r, $A = f(r)$, then $\frac{dA}{dr}$ is the rate of change of A wrt r.

Example 5

If $p = 3t^2 - 2t + 1$, *find* $\frac{dp}{dt}$.

p is a function of t so we take an increment δt in t.
The corresponding increment in p is δp.

$\frac{dp}{dt} = \lim_{\delta t \to 0} \frac{\delta p}{\delta t}$

$p + \delta p = 3(t + \delta t)^2 - 2(t + \delta t) + 1$

Now show that $\delta p = 6t\delta t + 3(\delta t)^2 - 2\delta t$

$\frac{\delta p}{\delta t} = 6t + 3\delta t - 2$ and the limiting value is $6t - 2$.

Hence $\frac{dp}{dt} = 6t - 2$.

NOTE ON INDICES

We shall be dealing with negative indices shortly so this note recalls the rules for indices.

To multiply powers of the same term, add the indices.

$x^3 \times x^2 = x \times x \times x \times x \times x = x^5 = x^{3+2}$

To divide powers of the same term, subtract the indices.

$x^5 \div x^2 = \frac{x \times x \times x \times x \times x}{x \times x} = x^3 = x^{5-2}$

$x \div x = 1 = x^{1-1} = x^0$ and $x^0 = 1$

$x \div x^3 = x^{1-3} = x^{-2}$ but $\frac{x}{x^3} = \frac{1}{x^2}$.

A negative index means the reciprocal: $x^{-n} = \frac{1}{x^n}$

THE DERIVATIVE OF ax^n

Here are listed some of the derivatives we have already found:

Function	*Derivative*
x^2	$2x$
$2x^2$	$4x$
$4x^3$	$12x^2$
$\frac{1}{x} = x^{-1}$	$-\frac{1}{x^2} = -x^{-2}$

Can you see a pattern in these?

The derivative of x^2 is $2x = \mathbf{2}x^{\mathbf{2}-1}$.

The derivative of $4x^3$ is $12x^2 = 4 \times \mathbf{3}x^{\mathbf{3}-1}$.

The derivative of x^{-1} is $-x^{-2} = \mathbf{-1}x^{\mathbf{-1}-1}$.

We deduce the following rule (which we shall not prove):

to differentiate a single term, multiply by the index and then reduce the index by 1.

For example, to find the derivative of $5x^3$:

the index 3 becomes a multiplier

$5 \times 3\, x^2$ ⟵ the new index is $3 - 1 = 2$

The derivative of ax^n is anx^{n-1}.

What about the derivative of a constant, say $y = 5$? This is 0, as the gradient is always 0.

The derivative of a constant is 0.

If $y = ax^n$, $\frac{dy}{dx} = anx^{n-1}$

If $y = k$ (a constant), $\frac{dy}{dx} = 0$

THE DERIVATIVE OF A POLYNOMIAL

In Example 5, we saw that the derivative of the polynomial $3t^2 - 2t + 1$ was $6t - 2$. The derivative of $3t^2$ is $6t$, the derivative of $-2t$ is -2 and the derivative of 1 is 0. These are added to obtain the derivative of the polynomial.

The derivative of a polynomial is the sum of the separate derivatives of the terms.

This rule applies only to *polynomials* and does not apply to functions such as $\sqrt{3x-1}$ or $\frac{x+2}{x-1}$ which are not polynomials.

Example 6

Differentiate wrt x (a) $3x^5 + 7$, (b) $x^3 - \frac{1}{2}x^2 - \frac{1}{x}$, (c) $(2x - 3)^2$, (d) $\frac{4}{x^3}$,
(e) $ax^3 + 2bx^2 - cx + 7$ where a, b and c are constants.

(a) If $y = 3x^5 + 7$, then $\frac{dy}{dx} = 3 \times 5x^{5-1} + 0 = 15x^4$.

Note: Do not write $3x^5 + 7 = 15x^4$. This is incorrect. Use a letter such as y for the function and then write $\frac{dy}{dx}$.

(b) If $y = x^3 - \frac{1}{2}x^2 - \frac{1}{x} = x^3 - \frac{1}{2}x^2 - x^{-1}$
then $\frac{dy}{dx} = 3x^2 - \frac{1}{2} \times 2x^1 - (-1)x^{-2} = 3x^2 - x + \frac{1}{x^2}$

Rewrite reciprocals such as $\frac{1}{x^2}$ in terms of negative indices before differentiating.

(c) Here we express $(2x - 3)^2$ as a polynomial first by expansion.
$(2x - 3)^2 = 4x^2 - 12x + 9$
If $y = 4x^2 - 12x + 9$, then $\frac{dy}{dx} = 8x - 12x^0 + 0 = 8x - 12$, as $x^0 = 1$.

(d) If $y = \frac{4}{x^3} = 4x^{-3}$, then $\frac{dy}{dx} = 4(-3)x^{-3-1} = -12x^{-4}$
which can be left in this form or written as $-\frac{12}{x^4}$.

(e) If $y = ax^3 + 2bx^2 - cx + 7$, then $\frac{dy}{dx} = 3ax^2 + 4bx - cx^0 + 0 = 3ax^2 + 4bx - c$.

Example 7

Find the coordinates of the points on the curve $y = x^3 - 3x^2 - 9x + 6$ where the gradient is 0.

The gradient = $\frac{dy}{dx}$.

$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$

If the gradient = 0, then $\frac{dy}{dx} = 0$ and so $3(x - 3)(x + 1) = 0$ which gives $x = 3$ or $x = -1$.

When $x = 3$, $y = 3^3 - 3(3^2) - 9(3) + 6 = -21$: coordinates $(3,-21)$

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 6 = 11$: coordinates $(-1,11)$

Example 8

(a) Differentiate $A = 2\pi r^2 + 2\pi rh$ wrt r, where h is a constant.

(b) If $T = \frac{(3p + 1)^2}{p}$, find $\frac{dT}{dp}$ and the values of p for $\frac{dT}{dp} = 0$.

(a) $\frac{dA}{dr} = 2\pi \times 2r^1 + 2\pi hr^0 = 4\pi r + 2\pi h$

(b) Here we rewrite the expression as a polynomial first.

$\frac{(3p+1)^2}{p} = \frac{9p^2+6p+1}{p} = 9p + 6 + \frac{1}{p}$ (dividing each term in

$= 9p + 6 + p^{-1}$ the numerator by p)

Then $\frac{dT}{dp} = 9 + 0 + (-1)p^{-2} = 9 - \frac{1}{p^2}$

If $\frac{dT}{dp} = 0$, then $9 = \frac{1}{p^2}$ and $p = \pm\ \frac{1}{3}$.

Exercise 9.4 *(Answers on page 627.)*

1 Differentiate wrt x:

(a) $5x$ (b) $4x^2$ (c) 7

(d) $3x^2 - 5$ (e) $3x^2 - x - 1$ (f) $x^3 - x^2 - x - 1$

(g) $1 - 3x^2$ (h) $(x-1)^2$ (i) $\frac{4}{x}$

(j) $3x^2 + \frac{1}{x}$ (k) $(x-2)^3$ (l) $(x - \frac{1}{x})^2$

(m) $\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2}$ (n) $\frac{x^3 - 2x^2 + x - 3}{x}$ (o) $\frac{(2x-1)^2}{x^2}$

(p) $\frac{2x^2 - x - 1}{x^3}$

2 Differentiate $s = 3t^2 - 4t - 2$ wrt t.

3 If $W = 3r^3 - 2r^2 + r - 3$, find $\frac{dW}{dr}$.

4 Given $u = (3t-2)^2$, find $\frac{du}{dt}$.

5 $y = 4x^2 - \mu x + 6$. Given that the rate of change of y wrt x is 2 when $x = 1$, find the value of μ.

6 Find the gradient on the curve $y = 4x^3 - 10x + 1$ at the point where $x = -1$.

7 The gradient on the curve $y = ax^2 - 3x + 4$ where $x = -2$ is 13. Find the value of a.

8 Find the coordinates of the points on the curve $y = 3x^3 - 4x + 2$ where the gradient is 0.

9 Find the coordinates of the points on the curve $y = x^3 - 3x^2 + x - 5$ where the gradient is 1.

10 Find the values of z for which $\frac{dP}{dz} = 0$ where $P = 4z^3 - 2z^2 - 8z + 5$.

11 Show that there is only one point on the curve $y = x^4 - 32x + 10$ where the gradient is 0 and find its coordinates.

12 Given that $u = 4t^3 + 3t^2 - 6t - 1$, find the values of t for which $\frac{du}{dt} = 12$.

13 Find the values of t for which $\frac{ds}{dt} = 0$ given that $s = 4t^3 + t^2 - 2t - 5$.

14 Given $v = 4s^2 - 12s - 7$, find $\frac{dv}{ds}$. For what value of s is $\frac{dv}{ds} = 4$?

15 Given that the curve $y = ax^2 + \frac{b}{x}$ has a gradient of 5 at the point (1,1), find the values of a and b. What is the gradient of the tangent to the curve at the point where $x = \frac{1}{2}$?

16 If the gradient on the curve $y = ax + \frac{b}{x}$ at the point (–1,–1) is 5, find the values of a and b.

17 The curve given by $y = ax^3 + bx^2 + 3x + 2$ passes through the point (1,2) and the gradient at that point is 7. Find the values of a and b.

18 Given $y = 2x^3 - 3x^2 - 12x + 5$, find the domain of x for which $\frac{dy}{dx} \geq 0$.

19 The function P is given by $P = \frac{a}{t} + bt^2$ and when $t = 1$, $P = -1$.
The rate of change of P when $t = \frac{1}{2}$ is –5. Find the values of a and b.

20 Given that $R = mp^4 + np^2 + 3$, find $\frac{dR}{dp}$. When $p = 1$, $\frac{dR}{dp} = 12$ and when $p = \frac{1}{4}$, $\frac{dR}{dp} = -\frac{3}{4}$. Find the values of m and n.

COMPOSITE FUNCTIONS

In part (c) of Example 6, to find the derivative of $(2x - 3)^2$, we first expand it into a polynomial. Similarly, if we want to find the derivative of $y = (3x - 2)^5$, we first expand it into a polynomial. This would be rather lengthy so we look for a neater method. To do this we take $(3x - 2)^5$ as a **composite** or **combined** function.

The function $y = (3x - 2)^5$ can be built up from two simpler functions, $u = 3x - 2$ and then $y = u^5$. We call u the **core** function.

Now u is a function of x so $\frac{du}{dx} = 3$. y is a function of the core u so $\frac{dy}{du} = 5u^4$. To obtain $\frac{dy}{dx}$ from these two derivatives we use a rule for the derivative of composite functions (which we shall not prove):

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

where y is a function of u and u is a function of x.

Note: du cannot be cancelled on the right hand side as these are not fractions but derivatives. However the notation suggests the result and is easy to remember.

Then $\frac{dy}{dx} = 5u^4 \times 3 = 15u^4 = 15(3x - 2)^4$.

Example 9

Find $\frac{dy}{dx}$ given that $y = (x^2 - 3x + 1)^4$.

Take $u = x^2 - 3x + 1$ as the core. Then $y = u^4$.

$\frac{dy}{du} = 4u^3$,differentiating y wrt the core u.

$\frac{du}{dx} = 2x - 3$, differentiating the core wrt x.

Multiply these two derivatives to obtain $\frac{dy}{dx}$:

$\frac{dy}{dx} = 4u^3 \times (2x - 3) = 4(2x - 3)(x^2 - 3x + 1)^3$.

Example 10

Find $\frac{dy}{dx}$ *if* $y = (ax^2 + bx + c)^n$.

Take $u = ax^2 + bx + c$ and then $y = u^n$.

$\frac{dy}{du} = nu^{n-1}$ and $\frac{du}{dx} = 2ax + b$.

Then $\frac{dy}{dx} = nu^{n-1} \times (2ax + b) = n(2ax + b)(ax^2 + bx + c)^{n-1}$.

With practice, $\frac{dy}{dx}$ can be written down in two steps on one line.

Suppose $y =$ (**core**)^n (**core** being a function of x)

Step 1 *Step 2*

$$\frac{dy}{dx} = n(\textbf{core})^{n-1} \times \frac{d(\textbf{core})}{dx}$$

derivative of (**core**)^n wrt core — derivative of **core** wrt x

Example 11

Differentiate $\frac{2}{x^2 - 3x + 1}$ *wrt x.*

Take $y = 2(\underbrace{x^2 - 3x + 1}_{\text{core}})^{-1}$

Then $\frac{dy}{dx} = 2(-1)(x^2 - 3x + 1)^{-2} \times (2x - 3)$ ← derivative of core wrt x

$$= \frac{-2(2x - 3)}{(x^2 - 3x + 1)^2}$$

Example 12

Given that $s = 3t - \frac{1}{1 - 2t}$, *find (a)* $\frac{ds}{dt}$ *and (b) the values of s when* $\frac{ds}{dt} = \frac{25}{9}$.

(a) $s = 3t - (\underbrace{1 - 2t}_{\text{core}})^{-1}$

$\frac{ds}{dt} = 3 - (-1)(1 - 2t)^{-2}(-2)$ ← derivative of core wrt t

$= 3 - \frac{2}{(1 - 2t)^2}$

(b) If $3 - \frac{2}{(1 - 2t)^2} = \frac{25}{9}$, then $\frac{2}{(1 - 2t)^2} = \frac{2}{9}$ i.e. $9 = (1 - 2t)^2$.

Hence $9 = 1 - 4t + 4t^2$ or $4t^2 - 4t - 8 = 0$

which gives $t^2 - t - 2 = 0$ or $(t - 2)(t + 1) = 0$ and $t = 2$ or -1.

When $t = 2$, $s = 6 - \frac{1}{1 - 4} = 6\frac{1}{3}$

and when $t = -1$, $s = -3 - \frac{1}{3} = -3\frac{1}{3}$.

Exercise 9.5 *(Answers on page 627.)*

1 Differentiate the following wrt x:

(a) $(x-3)^5$ (b) $(3x-1)^7$ (c) $(5-2x)^3$

(d) $(4x-5)^{10}$ (e) $(4x-3)^4$ (f) $(x^2-x+1)^3$

(g) $(3-x-2x^2)^5$ (h) $\frac{1}{x-2}$ (i) $\frac{4}{1-3x}$

(j) $\frac{4}{3+2x}$ (k) $(x-\frac{1}{x})^4$ (l) $\frac{1}{x^2+3}$

(m) $\frac{4}{x^2-x-1}$ (n) $(ax+b)^n$ (o) $\frac{2}{(2x-3)^4}$

(p) $\frac{1}{(1-3x-2x^2)^3}$ (q) $(2x-\frac{1}{2x})^3$

2 If $s=(2t-1)^3$, find (a) $\frac{ds}{dt}$ and (b) the value of t for which $\frac{ds}{dt}=24$.

3 If $v=(3t^2-2t+1)^2$ find the value of $\frac{dv}{dt}$ when $t=-1$.

4 Given that $A=\frac{t^2}{2}-\frac{(1-t)^3}{5}$, find $\frac{dA}{dt}$ and simplify the result. Hence find the value of t for which $\frac{dA}{dt}=1$.

5 If $s=\frac{3}{4-2r}$ find $\frac{ds}{dr}$ and the values of s when $\frac{ds}{dr}=\frac{1}{6}$.

6 The equation of a curve is $y=2x-\frac{4}{x+1}$. Find (a) $\frac{dy}{dx}$ and (b) the gradient of the curve when $x=-3$.

7 Find the gradient of the curve $y=\frac{3}{x^2-2x+1}$ where $x=2$.

8 If $y=\frac{1}{x+1}$, find the coordinates of the points where the gradient is $-\frac{1}{4}$.

9 If $y=\frac{1}{(x-3)^2}$, find the coordinates of the point where the gradient = 2.

10 Given that $v=\frac{3}{1-4t}$, find (a) $\frac{dv}{dt}$, (b) the values of t when $\frac{dv}{dt}=3$.

11 If $y=3t+1+\frac{1}{1+2t}$, find (a) $\frac{dy}{dt}$ and (b) the values of t when $\frac{dy}{dt}=2\frac{1}{3}$.

12 When $x=1$, the gradient of the curve $y=\frac{1}{3+ax}$ is 2. Find the values of a.

13 Given that $L=\frac{1}{a+bx}$ and that $L=1$ and $\frac{dL}{dx}=3$ when $x=1$, find the values of a and b.

14 The curve $y=\frac{1}{a+bx}$ passes through the point (1,–1) and its gradient at that point is 2. Find the values of a and b.

THE SECOND DIFFERENTIAL COEFFICIENT $\frac{d^2y}{dx^2}$

If y is a function of x, then $\frac{dy}{dx}$ is also a function of x (or a constant).

Hence we can differentiate $\frac{dy}{dx}$ wrt x. This gives the **second differential coefficient** $\frac{d(\frac{dy}{dx})}{dx}$ which is written as $\frac{d^2y}{dx^2}$ (read *dee two y by dee x two*) for brevity. The 2's are not squares but symbolize differentiating twice.

The square of $\frac{dy}{dx}$ is written as $(\frac{dy}{dx})^2$.

$\frac{d^2y}{dx^2}$ is sometimes also written as $f''(x)$ where $y=f(x)$.

Example 13

Find $\frac{d^2y}{dx^2}$ *and* $\left(\frac{dy}{dx}\right)^2$ *if*

(a) $y = 2x^3 - 3x^2 + 1$, *(b)* $y = (4x - 1)^3$, *(c)* $y = \frac{1}{2-3x}$.

(a) $\frac{dy}{dx} = 6x^2 - 6x$

$\frac{d^2y}{dx^2} = 12x - 6$

$\left(\frac{dy}{dx}\right)^2 = (6x^2 - 6x)^2$ which is quite different from $\frac{d^2y}{dx^2}$.

(b) $\frac{dy}{dx} = 3(4x - 1)^2 \times 4 = 12(4x - 1)^2$

$\frac{d^2y}{dx^2} = 24(4x - 1) \times 4 = 96(4x - 1)$

$\left(\frac{dy}{dx}\right)^2 = 144(4x - 1)^4$.

(c) $y = (2 - 3x)^{-1}$

$\frac{dy}{dx} = (-1)(2 - 3x)^{-2} \times (-3) = 3(2 - 3x)^{-2}$

$\frac{d^2y}{dx^2} = 3(-2)(2 - 3x)^{-3} \times (-3) = 18(2 - 3x)^{-3}$

$\left(\frac{dy}{dx}\right)^2 = \frac{9}{(2 - 3x)^4}$

As we shall see, $\frac{d^2y}{dx^2}$ has important applications. It is also possible to find further derivatives, such as $\frac{d^3y}{dx^3}$, etc. but we shall not use these.

Exercise 9.6 *(Answers on page 626.)*

1 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for

(a) $4x^3 - 5x^2 + 8$ (b) $(2x - 7)^3$ (c) $(1 - 4x)^4$ (d) $\frac{1}{x}$

(e) $x^2 - \frac{1}{x}$ (f) $\frac{3}{2-x}$ (g) $x^4 - x^2 - \frac{1}{x^2}$

2 If $s = 3t^2 - \frac{2}{t^2}$, find $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$.

3 If $y = (ax + 2)^2$ and $\frac{d^2y}{dx^2} = 18$, find the values of a.

4 $y = ax^3 + bx$. Given that $\frac{dy}{dx} = -1\frac{1}{4}$ and that $\frac{d^2y}{dx^2} = 3$ when $x = \frac{1}{2}$, find the values of a and b.

5 For the function $y = (ax + b)^2$, $\frac{dy}{dx} = -6$ when $x = \frac{1}{3}$ and $\frac{d^2y}{dx^2} = 18$. Find the values of a and b.

6 If the gradient of the curve $y = 2x^3 + \mu x^2 - 5$ is -2 when $x = 1$, find the value of μ and the value of $\frac{d^2y}{dx^2}$ at that point.

7 If $y = \frac{1}{2-x}$, find $(\frac{dy}{dx})^2$ and $\frac{d^2y}{dx^2}$. Show that $y\frac{d^2y}{dx^2} = 2(\frac{dy}{dx})^2$.

8 If $s = 3t^3 - 30t^2 + 36t + 2$, find the values of t for which $\frac{ds}{dt} = 0$ and the value of t for which $\frac{d^2s}{dt^2} = 0$.

9 If $y = 2x^3 - 4x^2 + 9x - 5$, what is the range of values of x for which $\frac{d^2y}{dx^2} \geq 0$?

SUMMARY

- If $y = f(x)$, $f'(x)$ is the gradient function. The value of $f'(x)$ is the gradient at a given point.
- $f'(x) = \frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$ where δx is the increment in x.
- $\frac{dy}{dx}$ is the derivative or differential coefficient of y wrt x. It measures the rate of change of y wrt x.
- If $y = ax^n$, $\frac{dy}{dx} = nax^{n-1}$; if $y = k$ (a constant), $\frac{dy}{dx} = 0$
- The derivative of a sum of terms or a polynomial is the sum of the derivatives of the separate terms.
- If $y = f(u)$ where u is a function of x, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
- $\frac{d(\frac{dy}{dx})}{dx}$ is the second differential coefficient of y wrt x, written as $\frac{d^2y}{dx^2}$ or as $f''(x)$ if $y = f(x)$.

REVISION EXERCISE 9 *(Answers on page 628.)*

1 Differentiate wrt x:

(a) $(x - 5)^3$ (b) $(1 - 2x)^5$

(c) $\frac{2}{1-4x}$ (d) $(2x^2 - 1)^3$

(e) $(1 - 3x - 2x^2)^3$ (f) $\frac{1}{x(1-x)}$

(g) $(2x - \frac{1}{x})^3$ (h) $\frac{x^3 - 2x^2 + 1}{2x^2}$

(i) $\frac{(x-1)(x+4)}{x}$ (j) $\frac{1-3x}{2} - \frac{2}{1-3x}$

2 If $y = x^3 - 3x^2 + 7$, for what range of values of x is $\frac{dy}{dx} \leq 0$?

3 Find the gradient of the curve $y = 5 + 2x - 3x^2$ at each of the points where it meets the x-axis.

4 (a) Find the gradient of the curve $y = \frac{6}{x}$ where $x = 3$.
Hence find (b) the equation of the tangent at that point and (c) the coordinates of the points where this tangent meets the axes.
(d) Calculate the distance between these points.

5 Given that $s = 3t^2 - 4t + 1$, find the rate of change of s wrt t when $s = 5$.

6 If $y = 4x - \frac{1}{1-2x}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

7 For the function $y = 2x^3 - 4x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Hence find the value of $\dfrac{\frac{dy}{dx}}{\sqrt{1 + \frac{d^2y}{dx^2}}}$ when $x = 2$.

8 If $A = \frac{1}{1 + \mu r}$ where μ is a constant > 0 and the rate of change of A wrt r is $-\frac{5}{9}$ when $r = 0.4$, find the value of μ.

9 The tangent at the point (a,b) on the curve $y = 1 - x - 2x^2$ has a gradient of 7. Find the values of a and b.

10 The curve $y = \frac{a}{x} + bx$ (a, b constants) passes through the points A(1,–1) and B(4,$-11\frac{1}{2}$). (a) Find the value of a and of b. (b) Show that the tangent to the curve at the point where $x = -2$ is parallel to AB.

11 (a) Show that the gradients of the tangents to the curve $y = x^2 - x - 2$ where the curve meets the x-axis are numerically equal.
(b) Find the equations of these tangents and show that they intersect on the axis of the curve.

12 The line $y = x + 1$ meets the curve $y = x^2 - x - 2$ at the points A and B. Find the gradients of the tangents to the curve at these points.

13 If $p = 2s^3 - s^2 - 28s$, find the values of s which make $\frac{dp}{ds} = 0$ and for these values of s find the value of $\frac{d^2p}{ds^2}$.

14 The gradient of the curve $y = ax^2 + bx + 2$ at the point (2,12) is 11. Find the values of a and b.

15 If $y = x^3 + 3x^2 - 9x + 2$, for what range of values of x is $\frac{dy}{dx}$ negative?

16 Given that $y = (x + 2)^2 - (x - 2)^3$, find the range of values of x for which $\frac{dy}{dx} \geq 0$.

17 If $y = \frac{A}{x} + Bx$, where A and B are constants, show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y$.

18 The velocity **v** of a moving body whose position vector **r** is given by $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, where x and y are functions of t, is the vector $\mathbf{v} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$ (Fig.9.10).

(a) If $\mathbf{r} = 2t\mathbf{i} + (5 - t)\mathbf{j}$, use the above to find the vector **v**.

(b) Find the value of t for which **v** is perpendicular to **r** .

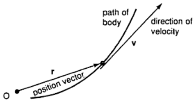

Fig.9.10