# Binomial Expansions

# 5

A **binomial** is an expression of two terms, such as $(x + y)$, $(a - b)$, etc. If the **binomial** $(a + b)$ is squared, the result is the **expansion** of $(a + b)^2$. Write down this expansion.

Now examine the pattern obtained if we expand $(a + b)^3$, $(a + b)^4$, etc.

$$(a + b)^3 = (a + b)^2(a + b)$$
$$= (a^2 + 2ab + b^2)(a + b)$$

To find this, multiply each term of $(a^2 + 2ab + b^2)$ by $a$, then by $b$ and add the results.

Multiplying by $a$: $a^3 + 2a^2b + ab^2$

Multiplying by $b$: $a^2b + 2ab^2 + b^3$

Adding: $a^3 + 3a^2b + 3ab^2 + b^3$

Note that the powers of $a$ and $b$ add up to 3 and that the coefficients are **1 3 3 1.**

Now find $(a + b)^4 = (a + b)^3(a + b)$ in the same way.

You should obtain $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

The powers add up to 4 and the coefficients are **1 4 6 4 1.**

Once more, find the expansion of $(a + b)^5$. Can you see the pattern?

| | |
|---|---|
| 1 2 1 | coefficients of $(a + b)^2$ |
| 1 3 3 1 | coefficients of $(a + b)^3$ |
| 1 4 6 4 1 | coefficients of $(a + b)^4$ |
| 1 5 10 10 5 1 | coefficients of $(a + b)^5$ |

Each line starts and ends with 1. Go along the $(a + b)^2$ line and add the coefficients in pairs. You will find that the sum of each pair gives the coefficient in the next line. Repeat for the other lines. Hence find the coefficients for $(a + b)^6$ and $(a + b)^7$.

Note that the coefficients are symmetrical and that the second coefficient is equal to the power of the expansion. For $(a + b)^n$ there are $(n + 1)$ terms, where $n$ is an integer. Make a copy of the triangle up to a power of 8 to keep for reference.

This pattern is called **Pascal's Triangle** after the French mathematician Pascal (1623 – 1662) but it was known in China long before his time. By working through the triangle we can find the coefficients for any power $n$ of $(a + b)$.

Later in this chapter, we will introduce the **Binomial Theorem** which gives a formula for the coefficients, but for most of our work the triangle will be sufficient.

### Example 1

*Expand $(a + b)^8$.*

From the triangle the coefficients are 1 8 28 56 70 56 28 8 1.

Then $(a + b)^8 = 1a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + 1b^8$

Note that the powers of $a$ decrease from 8 to 0 while the powers of $b$ increase from 0 to 8. The sum of these powers is always 8.

$(a + b)$ is the model binomial but we can replace $a$ or $b$ by other expressions.

### Example 2

*Expand $(2x - 1)^4$.*

The initial coefficients are 1 4 6 4 1. Here $a = 2x$, $b = -1$.

$$\begin{aligned}\text{Then } (2x - 1)^4 &= \mathbf{1}(2x)^4 + \mathbf{4}(2x)^3(-1) + \mathbf{6}(2x)^2(-1)^2 + \mathbf{4}(2x)(-1)^3 + \mathbf{1}(-1)^4 \\ &= 16x^4 - 32x^3 + 24x^2 - 8x + 1\end{aligned}$$

The coefficients are now quite different. The powers of $x$ are in **descending** order.

### Example 3

*Find in ascending powers of $x$ the expansion of $(2 - \frac{x}{2})^6$.*

The initial coefficients are 1 6 15 20 15 6 1. The expansion is

$$2^6 + \mathbf{6}(2^5)\left(-\tfrac{x}{2}\right) + \mathbf{15}(2^4)\left(-\tfrac{x}{2}\right)^2 + \mathbf{20}(2^3)\left(-\tfrac{x}{2}\right)^3 + \mathbf{15}(2^2)\left(-\tfrac{x}{2}\right)^4 + \mathbf{6}(2)\left(-\tfrac{x}{2}\right)^5 + \left(-\tfrac{x}{2}\right)^6$$

$$= 64 - 6(2^4)x + 15(2^2)x^2 - 20x^3 + 15\left(\tfrac{x^4}{2^2}\right) - 6\left(\tfrac{x^5}{2^4}\right) + \tfrac{x^6}{2^6}$$

$$= 64 - 96x + 60x^2 - 20x^3 + \tfrac{15x^4}{4} - \tfrac{3x^5}{8} + \tfrac{x^6}{64}$$

## Exercise 5.1 *(Answers on page 618.)*

**1** Find, in descending powers of $x$, the expansions of:

(a) $(x - 2)^4$ (b) $(2x - 3)^3$ (c) $(2x + 1)^5$

(d) $\left(x - \frac{1}{2}\right)^5$ (e) $\left(x + \frac{1}{x}\right)^6$ (f) $\left(\frac{x}{4} - 2\right)^4$

**2** Expand, in ascending powers of $x$:

(a) $(1 - 2x)^5$ (b) $(2 - 3x)^4$ (c) $\left(2 - \frac{x}{2}\right)^6$ (d) $(1 - x^2)^3$

**3** Find, in ascending powers of $x$, the first four terms in the expansion of:

(a) $(2 - x)^5$ (b) $(1 - 2x)^7$ (c) $\left(1 - \frac{x}{2}\right)^8$ (d) $\left(4 - \frac{x}{2}\right)^5$

**4** Find the expansions of (a) $(3x - 2y)^4$ (b) $\left(x - \frac{1}{x}\right)^5$.

**5** Expand $(a + b)^5$. If $a = \frac{3}{4}$ and $b = \frac{1}{4}$, find the value (as a fraction) of the fourth term of the expansion.

**6** Write down the first four terms of the expansion of $(1 - x)^6$ in ascending powers of $x$. Using these terms, find an approximate value of $(0.99)^6$.

**7** (a) Write down the expansions of $(1 + x)^3$ and $(1 - x)^3$.
(b) Hence simplify $(1 + x)^3 + (1 - x)^3$. Use your result to find the exact value of $(1 + \sqrt{2})^3 + (1 - \sqrt{2})^3$.

**8** By using the expansions of $(2 + x)^4$ and $(2 - x)^4$, find the exact value of $(2 + \sqrt{3})^4 + (2 - \sqrt{3})^4$.

**9** (a) Write down the expansions of $(1 + x)^4$ and $(1 - x)^4$.
(b) Hence simplify the expression $(1 + x)^4 - (1 - x)^4$. Use your result to find the value of $1.01^4 - 0.99^4$.

**10** (a) Obtain the expansions of $\left(x + \frac{1}{x}\right)^5$ and $\left(x - \frac{1}{x}\right)^5$.
(b) Hence simplify $\left(x + \frac{1}{x}\right)^5 - \left(x - \frac{1}{x}\right)^5$.
(c) Choosing a suitable value of $x$, find the value of $2.5^5 - 1.5^5$.

### Example 4

*(a) Expand $(1 + y)^4$ in ascending powers of $y$.*
*(b) Hence find the expansion of $(1 + x - x^2)^4$ as far as the term in $x^3$.*

(a) $(1 + y)^4 = 1 + 4y + 6y^2 + 4y^3 + y^4$

(b) Now substitute $(x - x^2)$ for $y$.

$$\begin{aligned}(1 + x - x^2)^4 &= 1 + 4(x - x^2) + 6(x - x^2)^2 + 4(x - x^2)^3 + (x - x^2)^4 \\ &= 1 + 4x - 4x^2 + 6(x^2 - 2x^3\ldots) + 4(x^3\ldots) + \ldots \\ &\qquad \text{(where we do not keep any terms higher than } x^3\text{)} \\ &= 1 + 4x - 4x^2 + 6x^2 - 12x^3 + 4x^3 \quad \text{(up to the term in } x^3\text{)} \\ &= 1 + 4x + 2x^2 - 8x^3 \quad \text{(up to the term in } x^3\text{)}\end{aligned}$$

### Example 5

*(a) Find, in ascending powers of $x$, the expansions of $(1 - 2x)^3$ and $(2 + x)^4$.*
*(b) Hence find the first four terms of the expansion of $(1 - 2x)^3(2 + x)^4$.*

(a)
$$\begin{aligned}(1 - 2x)^3 &= 1 + 3(-2x) + 3(-2x)^2 + (-2x)^3 \\ &= 1 - 6x + 12x^2 - 8x^3\end{aligned}$$

$$\begin{aligned}(2 + x)^4 &= 2^4 + 4(2^3)(x) + 6(2^2)(x^2) + 4(2)(x^3) + x^4 \\ &= 16 + 32x + 24x^2 + 8x^3 + x^4\end{aligned}$$

(b) $(1-2x)^3(2+x)^4 = (1-6x+12x^2-8x^3)(16+32x+24x^2+8x^3+x^4)$

The first four terms will go up to the power of $x^3$. So we multiply the terms in the first bracket by 16, $32x$, $24x^2$ and $8x^3$ and leave out any terms higher than $x^3$.

| | |
|---|---|
| Multiplying by 16 | $16-96x+192x^2-128x^3$ |
| Multiplying by $32x$ | $32x-192x^2+384x^3$ |
| Multiplying by $24x^2$ | $24x^2-144x^3$ |
| Multiplying by $8x^3$ | $8x^3$ |
| Adding | $16-64x+24x^2+120x^3$ |

## Example 6

*(a) Find the terms in $x^3$ and $x^4$ in the expansion of $(3-\frac{x}{3})^6$ in ascending powers of x.*

*(b) Hence find the coefficient of $x^4$ in the expansion of $(1-\frac{x}{2})(3-\frac{x}{3})^6$.*

(a) $(3-\frac{x}{3})^6 = 3^6 + 6(3^5)(-\frac{x}{3}) + 15(3^4)(-\frac{x}{3})^2 + 20(3^3)(-\frac{x}{3})^3 + 15(3^2)(-\frac{x}{3})^4 \ldots$

So the $x^3$ term is $-20x^3$ and the $x^4$ term is $+\frac{5x^4}{3}$.

(b) Then $(1-\frac{x}{2})(3-\frac{x}{3})^6 = (1-\frac{x}{2})(\ldots-20x^3+\frac{5x^4}{3}\ldots)$

The term in $x^4$ is found by multiplying the relevant terms as shown, and is $10x^4 + \frac{5x^4}{3}$ giving a coefficient of $\frac{35}{3}$.

## Example 7

*Write down and simplify the first three terms in the expansions (in ascending powers of x) of (a) $(1-\frac{3x}{2})^5$ and (b) $(2+x)^5$.*

*Hence find the coefficient of $x^2$ in the expansion of $(2-2x-\frac{3x^2}{2})^5$.*

(a) $(1-\frac{3x}{2})^5 = 1 + 5(-\frac{3x}{2}) + 10(-\frac{3x}{2})^2 \ldots = 1 - \frac{15x}{2} + \frac{45x^2}{2}$

(b) $(2+x)^5 = 2^5 + 5(2^4)(x) + 10(2^3)(x^2) \ldots = 32 + 80x + 80x^2$

We notice that $(2-2x-\frac{3x^2}{2})^5$ is the product of (a) and (b)

$$= [(1-\tfrac{3x}{2})(2+x)]^5$$

$$= [1-\tfrac{15x}{2}+\tfrac{45x^2}{2}\ldots][32+80x+80x^2\ldots]$$

The term in $x^2$ will be the sum of the products linked together, so the coefficient of $x^2$ is $80 - (\frac{15}{2}\times 80) + (\frac{45}{2}\times 32) = 200$.

### *Example 8*

*Find, in ascending powers of x, the first three terms in the expansions of*
*(a)* $(1+2x)^5$ *and (b)* $(1+px)^4$.
*(c) If the coefficient of* $x^2$ *in the expansion of* $(1+2x)^5(1+px)^4$ *is* $-26$*, find the value of p.*

(a) The first three terms of $(1+2x)^5$ are $1+5(2x)+10(2x)^2 = 1+10x+40x^2$.

(b) The first three terms of $(1+px)^4$ are $1+4(px)+6(px)^2 = 1+4px+6p^2x^2$.

(c) $(1+2x)^5(1+px)^4 = (1+10x+40x^2)(1+4px+6p^2x^2)$

We only require the term in $x^2$ so we pick out the terms (linked together above) whose products produce $x^2$:

$$1 \times 6p^2x^2 = 6p^2x^2$$
$$10x \times 4px = 40px^2$$
$$40x^2 \times 1 = 40x^2$$

giving $(6p^2+40p+40)x^2$.
Hence $6p^2+40p+40 = -26$
i.e. $3p^2+20p+33 = 0$ or $(3p+11)(p+3) = 0$
and so $p = -\frac{11}{3}$ or $-3$.

### *Example 9*

*(a) Find the first three terms in the expansion of* $(1-3x)^5$ *in ascending powers of x.*
*(b) If the first three terms in the expansion of* $(p+qx)(1-3x)^5$ *are* $3+rx+300x^2$*, state the value of p and find the values of q and r.*

(a) The first three terms of $(1-3x)^5$ are $1+5(-3x)+10(-3x)^2 = 1-15x+90x^2$.

(b) The first three terms of $(p+qx)(1-3x)^5$ will come from $(p+qx)(1-15x+90x^2)$.
The first term is $p$ so $p = 3$.
The term in $x$ is $qx-15px$ so $q-15p = r$ (i)
The term in $x^2$ is $90px^2-15qx^2$ so $90p-15q = 300$ (ii)

From (ii), $270-15q = 300$ so $q = -2$.
From (i), $r = -2-45 = -47$.

## Example 10

*(a) Expand $(1 - \frac{x}{2})^4$ in ascending powers of x.*
*(b) If this expansion is used to find the exact value of $(0.995)^4$, what value should be taken for x?*
*(c) Using this value, find $(0.995)^4$.*

(a) The coefficients in the expansion of $(a + b)^4$ are 1 4 6 4 1 and, in this case, $a = 1, b = -\frac{x}{2}$.
All the powers of $a = 1$.
Then $(1 - \frac{x}{2})^4 = 1 + 4(-\frac{x}{2}) + 6(-\frac{x}{2})^2 + 4(-\frac{x}{2})^3 + (-\frac{x}{2})^4$
$= 1 - 2x + \frac{3x^2}{2} - \frac{x^3}{2} + \frac{x^4}{16}$

(b) If $1 - \frac{x}{2} = 0.995$, then $\frac{x}{2} = 0.005$ and $x = 0.01$.

(c) Substitute $x = 0.01$ in the expansion.
$(0.995)^4 = 1 - 2(0.01) + \frac{3(0.01)^2}{2} - \frac{(0.01)^3}{2} + \frac{(0.01)^4}{16}$
Writing the positive and negative terms separately:

| *positive* | *negative* |
|---|---|
| $1$ | $-2(0.01) = -0.02$ |
| $\frac{3(0.01)^2}{2} = 0.000\,15$ | $-\frac{(0.01)^3}{2} = -0.000\,000\,5$ |
| $\frac{(0.01)^4}{16} = 0.000\,000\,000\,625$ | |
| 1.000 150 000 625 | − 0.020 000 5 |

which gives a sum of

$$\begin{array}{r} 1.000\,150\,000\,625 \\ -\,0.020\,000\,5 \\ \hline 0.980\,149\,500\,625 \end{array}$$

This is the exact value of $(0.995)^4$.
Compare this value with that obtained by using a calculator.

## Exercise 5.2 *(Answers on page 619.)*

1 Write down the expansion of $(1 - x)^4$. Use your result to find the expansion of $(1 - x + \frac{x^2}{2})^4$ in ascending powers of $x$ as far as the term in $x^2$.

2 Use the expansion of $(1 + x)^3$ to find the first three terms in the expansion of $(1 + \frac{x}{2} - x^2)^3$ in ascending powers of $x$.

3 Find the first three terms in the expansions in ascending powers of $x$ of (a) $(2 - x)^4$ and (b) $(3 - \frac{x}{2})^4$. Hence find the coefficients of $x$ and $x^2$ in the expansion of $(6 - 4x + \frac{x^2}{2})^4$.

4 (a) Write down the expansion of $(1 + x)^5$ in ascending powers of $x$ as far as the term in $x^3$.
(b) Hence find the first four terms in the expansion of $(1 + x - x^2)^5$.

**5** Expand in ascending powers of $x$, (a) $(1 + 2x)^4$ and (b) $(1 - x)^3$.
Hence find the first three terms in the expansion of $(1 + 2x)^4(1 - x)^3$.

**6** Write down the expansions of $(1 + 2x)^3$ and $(2 - \frac{x}{2})^4$ in ascending powers of $x$. Hence find the coefficient of the term in $x^2$ in the expansion of $(1 + 2x)^3(2 - \frac{x}{2})^4$.

**7** Find the coefficient of $x^3$ in the expansion of $(1 - 2x)^3(1 + \frac{3x}{2})^4$.

**8** Expand each of the binomials $(1 + x)^5$ and $(2 - x)^5$ as far as the term in $x^3$. Hence find the coefficient of $x^3$ in the expansion of $(2 + x - x^2)^5$.

**9** In the expansion of $(a + bx)^4$ in ascending powers of $x$, the first two terms are $16 - 96x$. Find the values of $a$ and $b$.

**10** The coefficient of the third term in the expansion of $(ax - \frac{1}{x})^5$ in descending powers of $x$ is 80. Find the value of $a$.

**11** (a) Expand $(1 + ax)^3$ and $(b + x)^4$ in ascending powers of $x$.
(b) If the first two terms in the expansion of $(1 + ax)^3(b + x)^4$ are $16 - 64x$, state the value of $b$, where $b > 0$, and find the value of $a$.

**12** In the expansion of $(p + qx)^4$ in ascending powers of $x$, the first two terms are $16 - \frac{8x}{3}$. Find the values of $p$ $(> 0)$ and $q$. Hence find the third term in the expansion.

**13** (a) Expand $(1 + px)^4$ and $(1 + qx)^3$ as far as the terms in $x^2$.
(b) Given that the coefficient of $x^2$ in the expansion of $(1 + px)^4(1 + qx)^3$ is $-6$ and that $p + q = 1$, find the values of $p$ and $q$.

**14** (a) State the expansions of (i) $(1 + ax)^3$ and (ii) $(1 + bx)^4$ in ascending powers of $x$.
(b) If the second and third terms in the expansion of $(1 + ax)^3(1 + bx)^4$ are $5x$ and $3x^2$ respectively, find the values of $a$ and $b$.

**15** (a) Find the coefficients of $x^4$ and $x^5$ in the expansion of $(2x - \frac{1}{2})^7$.
(b) Hence find the coefficient of $x^5$ in the expansion of $(\frac{x}{5} - 2)(2x - \frac{1}{2})^7$.

**16** Find the coefficients of $x^3$ and $x^4$ in the expansion of $(\frac{2}{3} - x)^6$.
Hence find the coefficient of $x^4$ in the expansion of $(1 + 3x)(\frac{2}{3} - x)^6$.

**17** Write down
(a) the first four terms in the expansion of $(1 - 2x)^4$, and
(b) the first three terms in the expansion of $(1 - x)^8$.
If the sum of the terms in (a) equals the sum of the terms in (b) where $x \neq 0$, find the value of $x$.

**18** State the first three terms in the expansion of $(1 + x)^4$ and hence find the first three terms in the expansion of $(1 + ax + bx^2)^4$. If these are $1 + 8x + 12x^2$, find the values of $a$ and $b$.

**19** If the expansion of $(1 - x - x^2)^{10}$ is used to find the value of $(0.89)^{10}$, what value of $x$ should be substituted?

**20** Write down the first three terms in the expansion of $(1 - x)^8$ in ascending powers of $x$. Use this expansion to find the value of $(0.999)^8$ correct to 5 significant figures.

# THE BINOMIAL THEOREM

The expansion of $(a + b)^n$ is given in full by a formula known as the **Binomial Theorem.** The formula is as follows:

$$(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + ... + \binom{n}{r} a^{n-r}b^r + ... + b^n$$

where $\binom{n}{r} = \frac{n(n-1)(n-2)...(n-r+1)}{r!}$ , and

$r!$ is **factorial *r*** and $r! = 1 \times 2 \times 3 \times 4 \times ... \times r$.

For example, $2! = 1 \times 2$, $3! = 1 \times 2 \times 3$, $5! = 1 \times 2 \times 3 \times 4 \times 5$ etc.

Hence $\binom{n}{1} = \frac{n}{1}$, $\binom{n}{2} = \frac{n(n-1)}{1 \times 2}$, $\binom{n}{3} = \frac{n(n-1)(n-2)}{1 \times 2 \times 3}$ and so on.

There are $(n + 1)$ terms in the expansion of $(a + b)^n$. The coefficients of the expansion are

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | $\binom{n}{1}$ | $\binom{n}{2}$ | ... | $\binom{n}{r}$ | ... | 1 |
| *term* | 1st | 2nd | 3rd | | $(r + 1)$th | | $(n + 1)$th |

The first and last coefficients are always 1 when $n$ is a positive integer (which it always will be in our work). Note that the coefficient for the $(r + 1)$th term is $\binom{n}{r}$.

Some calculators give the numerical value of $\binom{n}{r}$ (shown as ${}^nC_r$) but the formula needs to be known for algebraic terms.

### *Example 11*

*Show that* $\binom{10}{3} = \binom{10}{7}$.

$\binom{10}{3} = \frac{10 \times 9 \times 8}{1 \times 2 \times 3} = 120$

$\binom{10}{7} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 120$ after cancelling $4 \times 5 \times 6 \times 7$.

We could have expected this as the coefficients of $(a + b)^{10}$ are symmetrical.

This is an example of a general rule: $\binom{n}{r} = \binom{n}{n-r}$

Hence to find, say $\binom{12}{8}$ it is easier and quicker to find $\binom{12}{4}$.

### *Example 12*

*Using the theorem, find the coefficients in the expansion of* $(a + b)^7$.

The coefficients are $1, \binom{7}{1}, \binom{7}{2}, \binom{7}{3}, \binom{7}{4}, \binom{7}{5}, \binom{7}{6}$ and 1.

$\binom{7}{1} = \frac{7}{1} = 7$; $\binom{7}{2} = \frac{7 \times 6}{1 \times 2} = 21$; $\binom{7}{3} = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$;

$\binom{7}{4} = \binom{7}{3} = 35$; $\binom{7}{5} = \binom{7}{2} = 21$; $\binom{7}{6} = \binom{7}{1} = 7$

So the coefficients are 1, 7, 21, 35, 35, 21, 7 and 1 as we have found from Pascal's triangle.

Pascal's Triangle is the easier and quicker way of finding coefficients provided $n$ is not too large. If $n$ is large or is not known, then the Binomial Theorem must be used. The theorem is essential in more advanced work when $n$ may not be a positive integer.

### Example 13

*Find the first four terms in the expansion of $(x - 2)^{12}$.*

Here $a = x$, $b = -2$ and $n = 12$.

The first four terms are

$$x^{12} + \binom{12}{1}x^{11}(-2) + \binom{12}{2}x^{10}(-2)^2 + \binom{12}{3}x^9(-2)^3$$

$$= x^{12} + \frac{12}{1}x^{11}(-2) + \frac{12 \times 11}{1 \times 2}x^{10}(4) + \frac{12 \times 11 \times 10}{1 \times 2 \times 3}x^9(-8)$$

$$= x^{12} - 24x^{11} + 264x^{10} - 1760x^9$$

### Example 14

*Find the 5th and 6th terms in the expansion of $(2x - \frac{1}{2})^{10}$.*

Here $a = 2x$, $b = -\frac{1}{2}$ and $n = 10$.

The 5th term is given by $r = 4$ and the 6th term by $r = 5$.

The 5th term $= \binom{10}{4}(2x)^{10-4}(-\frac{1}{2})^4 = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4}(2x)^6(\frac{1}{2})^4 = 840x^6$.

Verify that the 6th term $= \frac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5}(2x)^5(-\frac{1}{2})^5 = -252x^5$.

### Example 15

*Write down (without simplifying) the first three terms in the expansion of $(x + b)^n$ where $n$ is a positive integer. If the coefficients of the second and third terms are $-8$ and 30 respectively, find the values of $n$ and $b$.*

$(x + b)^n = x^n + \binom{n}{1}x^{n-1}b + \binom{n}{2}x^{n-2}b^2$

Hence the coefficients of the second and third terms are $nb$ and $\frac{n(n-1)}{1 \times 2}b^2$ respectively.

Then $\quad nb = -8 \quad$ (i)

and $\quad \frac{n(n-1)}{2}b^2 = 30$ i.e. $n(n-1)b^2 = 60 \quad$ (ii)

We solve these equations for $n$ and $b$.

In (ii), substitute $b = \frac{-8}{n}$,

$$n(n-1)\frac{64}{n^2} = 60 \quad \text{or} \quad \frac{n-1}{n} = \frac{60}{64}$$

Then $64n - 64 = 60n$ from which we find $n = 16$.

From (i), $b = \frac{-8}{16} = -\frac{1}{2}$.

### *Example 16*

*Find the term independent of x in the expansion of* $(2x - \frac{1}{x})^{10}$.

From the theorem, the **$(r + 1)$th** term is

$\binom{10}{r}(2x)^{10-r}(-\frac{1}{x})^r = \binom{10}{r}2^{10-r}x^{10-r}(-1)^r(\frac{1}{x})^r$.

If this term is to be independent of $x$, then the $x$'s must cancel i.e. the powers of $x$ in the numerator and denominator must be equal.

Then $10 - r = r$ or $r = 5$.

So the **6th** term is independent of $x$.

This term is therefore $\binom{10}{5}2^5(-1)^5 = \frac{10 \times 9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4 \times 5} \times (-32) = -8064$.

## Exercise 5.3 *(Answers on page 619.)*

**1** Find the value of (a) $3!$, (b) $4!$, (c) $\frac{9!}{6!}$ (d) $\frac{12!}{4! \times 8!}$.

**2** Find the value of (a) $\binom{6}{2}$, (b) $\binom{9}{2}$, (c) $\binom{12}{8}$, (d) $\binom{15}{12}$.

**3** What is the value of $x$ if $\binom{11}{3x} = \binom{11}{x^2 - 7}$?

**4** Write down and simplify the first three terms of (a) $(1 + x)^{10}$, (b) $(x - \frac{1}{2})^{12}$, (c) $(x - \frac{1}{x})^9$.

**5** For the following expansions, find
(a) the coefficient of the ninth term in $(2x - 1)^{12}$;
(b) the coefficient of the fourth term in $(1 - 3x)^{10}$;
(c) the coefficient of the fifth term in $(x - \frac{1}{x})^9$.

**6** The coefficient of the second term in the expansion of $(1 + 2x)^n$ in ascending powers of $x$ is 40. Find the value of $n$.

**7** If the first three terms in the expansion of $(1 + ax)^n$ in ascending powers of $x$ are $1 + 6x + 16x^2$, find the values of $n$ and $a$.

**8** In the expansion of $(1 + px)^n$ in ascending powers of $x$, the second term is $18x$ and the third term is $135x^2$. Find the values of $n$ and $p$.

**9** Find the term independent of $x$ in the expansion of $(x - \frac{1}{2x^2})^9$.

**10** If the ratio of the 5th to the 6th term in the expansion of $(a + \frac{1}{x})^{11}$ is $5x : 1$, find the value of $a$.

## SUMMARY

- The coefficients in the expansion of $(a+b)^n$, where $n$ is a positive integer can be found from Pascal's Triangle:

| | |
|---|---|
| 1 2 1 | $(a+b)^2$ |
| 1 3 3 1 | $(a+b)^3$ |
| 1 4 6 4 1 | $(a+b)^4$ |
| 1 5 10 10 5 1 | $(a+b)^5$ |
| etc. | |

- The powers of $a$ decrease from $n$ to 0, the powers of $b$ increase from 0 to $n$. The sum of these powers is always $n$.
- Alternatively, the expansion of $(a+b)^n$ can be found using the Binomial Theorem, where $n$ is a positive integer:

  $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$

  where $\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$, and $r! = 1 \times 2 \times 3 \times \ldots \times r$.

## REVISION EXERCISE 5 *(Answers on page 619.)*

**A**

**1** Find, in ascending powers of $x$, the first four terms in the expansion of (i) $(1-3x)^5$, (ii) $(1+5x)^7$. Hence find the coefficient of $x^2$ in the expansion of $(1-3x)^5(1+5x)^7$. **(C)**

**2** Obtain the first three terms in the expansion of $(a+\frac{x}{b})^6$ in ascending powers of $x$. If the first and third terms are 64 and $\frac{80x^2}{3}$ respectively, find the values of $a$ and $b$ and the second term.

**3** Find the first three terms in the expansion of $(1-2x)^5$ in ascending powers of $x$, simplifying the coefficients.

Given that the first three terms in the expansion of $(a+bx)(1-2x)^5$ are $2+cx+10x^2$, state the value of $a$ and hence find the value of $b$ and of $c$. **(C)**

**4** (a) Expand $(1+2x)^5$ and $(1-2x)^5$ in ascending powers of $x$.
(b) Hence reduce $(1+2x)^5-(1-2x)^5$ to its simplest form.
(c) Using this result, evaluate $(1.002)^5-(0.998)^5$.

**5** Find, in ascending powers of $t$, the first three terms in the expansions of (i) $(1+\alpha t)^5$, (ii) $(1-\beta t)^8$. Hence find, in terms of $\alpha$ and $\beta$, the coefficient of $t^2$ in the expansion of $(1+\alpha t)^5(1-\beta t)^8$. **(C)**

**6** The first three terms in the expansion of $(1+\frac{x}{p})^n$ in ascending powers of $x$ are $1+x+\frac{9x^2}{20}$. Find the values of $n$ and $p$.

7 Write down and simplify the expansion of $(1 - p)^5$. Use this result to find the expansion of $(1 - x - x^2)^5$ in ascending powers of $x$ as far as the term in $x^3$. Find the value of $x$ which would enable you to estimate $(0.9899)^5$ from this expansion. (C)

8 Find which term is independent of $x$ in the expansion of $(x - \frac{1}{3x^2})^{15}$.

9 Obtain and simplify
(i) the first four terms in the expansion of $(2 + x^2)^6$ in ascending powers of $x$,
(ii) the coefficient of $x^4$ in the expansion of $(1 - x^2)(2 + x^2)^6$. (C)

10 In the expansion of $(1 - x)^{10}$, the sum of the first three terms is $\frac{4}{5}$ when a certain value of $x$ is substituted. Find this value of $x$.

11 Evaluate the coefficients of $x^5$ and $x^4$ in the binomial expansion of $(\frac{x}{3} - 3)^7$. Hence evaluate the coefficient of $x^5$ in the expansion of $(\frac{x}{3} - 3)^7(x + 6)$. (C)

12 If the first three terms in the expansion of $(1 + kx)^n$ in ascending powers of $x$ are $1 - 6x + \frac{33x^2}{2}$, find the values of $k$ and $n$.

13 Find, in ascending powers of $x$, the first three terms in the expansion of $(1 + ax)^6$. Given that the first two non-zero terms in the expansion of $(1 + bx)(1 + ax)^6$ are 1 and $\frac{-21x^2}{4}$, find the possible value of $a$ and of $b$. (C)

14 Find the ratio of the 6th term to the 8th term in the expansion of $(2x + 3)^{11}$ when $x = 3$.

15 In the expansion of $(1 + px)(1 + qx)^4$ in ascending powers of $x$, the coefficient of the $x$ term is $-5$ and there is no $x^2$ term. Find the value of $p$ and of $q$.

16 If the fifth term in the expansion of $(x + \frac{1}{x})^n$ is independent of $x$ find the value of $n$.

**B**

17 In the expansion of $(x^2 + \frac{2}{x^3})^7$, find which term will have the form $\frac{A}{x}$ where $A$ is an integer. Hence find the value of $A$.

18 The first three terms in the expansion of $(1 + x + ax^2)^n$ are $1 + 7x + 14x^2$. Find the values of $n$ and $a$.

19 (a) Obtain the expansions of $(1 + x)^5$ and $(1 + x^2)^5$ in ascending powers of $x$.
(b) Show that $(1 + x)(1 + x^2) = 1 + x + x^2 + x^3$.
(c) Hence find the first four terms in the expansion of $(1 + x + x^2 + x^3)^5$ in ascending powers of $x$.

20 For what value of $x$ is the fifth term of $(1 + 2x)^{10}$ equal to the sixth term of $(2 + x)^8$?

21 Show that (a) $(x - \frac{1}{x})^3 = x^3 - \frac{1}{x^3} - 3(x - \frac{1}{x})$ and
(b) $(x - \frac{1}{x})^5 = x^5 - \frac{1}{x^5} - 5(x^3 - \frac{1}{x^3}) + 10(x - \frac{1}{x})$.
Hence show that $x^5 - \frac{1}{x^5} = p^5 + 5p^3 + 5p$ where $p = x - \frac{1}{x}$.