Simultaneous Equations

2

Two linear equations, say $3x + 4y = -5$ and $2x - 3y = 8$, can be solved to find values of x and y which satisfy **both** equations simultaneously. As we have seen, this solution gives the coordinates of the point of intersection of the two lines represented by the equations.

In this Chapter we consider two simultaneous equations where one of them is not a linear equation but is an equation of the second degree such as $xy = 8$ or $x^2 + y^2 = 10$, etc. These are the equations of *curves*.

Example 1

Solve the following equations:

$$x + y = 9 \qquad (i)$$
$$xy = 8 \qquad (ii)$$

Equation (i) represents a straight line but equation (ii) is the equation of a **hyperbola,** a curve with two branches (Fig.2.1).

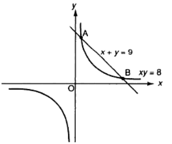


Fig.2.1

The line meets the curve at two different points (A and B) so we expect to obtain **two** solutions, giving the coordinates of A and B.

The usual method is to eliminate one of the variables. Make one variable the subject of the linear equation and then substitute this in the other (non-linear) equation. This will lead to a quadratic equation, which can usually be solved by factorization.

From (i), $x = 9 - y$.

Then substituting for x in (ii),

$$(9 - y)y = 8$$

i.e. $y^2 - 9y + 8 = 0$ or $(y - 8)(y - 1) = 0$.

Hence $y = 8$ or 1.

Now find the corresponding values of x from (i).

When $y = 8$, $x = 1$;
when $y = 1$, $x = 8$.

So the solutions are $x = 1$, $y = 8$ (coordinates of A)
or $x = 8$, $y = 1$ (coordinates of B).

Example 2

Find the coordinates of the points where the line

$$2x + 3y = -1 \qquad (i)$$

meets the curve

$$x(x - y) = 2 \qquad (ii)$$

We use the same method but the algebra will be more complicated as neither x nor y in (i) has a coefficient of 1.

Choosing y as the subject, we obtain from (i)

$$y = \frac{-1 - 2x}{3}.$$

Then substituting for y in (ii),

$$x\left(x - \frac{-1 - 2x}{3}\right) = 2 \quad \text{or} \quad x\left(\frac{3x + 1 + 2x}{3}\right) = 2$$

which simplifies to $x(5x + 1) = 6$ or $5x^2 + x - 6 = 0$.

Hence $(5x + 6)(x - 1) = 0$ giving $x = -\frac{6}{5}$ or 1.

From (i), when $x = -\frac{6}{5}$, $-\frac{12}{5} + 3y = -1$ so $y = \frac{7}{15}$,

and when $x = 1$, $2 + 3y = -1$ so $y = -1$.

Hence the coordinates of the two points are $(-\frac{6}{5}, \frac{7}{15})$ and $(1, -1)$.

Example 3

Solve the equations

$$x + 2y = 7 \quad (i)$$

and $$x^2 - 4x + y^2 = 1 \quad (ii)$$

What is the geometrical meaning of your answer?

From (i), choosing x as the subject for simplicity, $x = 7 - 2y$.

Then substituting for x in (ii),

$$(7 - 2y)^2 - 4(7 - 2y) + y^2 = 1$$

i.e. $49 - 28y + 4y^2 - 28 + 8y + y^2 = 1$

which reduces to $y^2 - 4y + 4 = 0$

or $(y - 2)(y - 2) = 0$ giving $y = 2$ or 2.

Then $x = 3$ or 3.

We obtain two *equal* solutions. This means that the line is a tangent to the curve. It *touches* the curve, which is a circle, at the point (3,2) as shown in Fig.2.2.

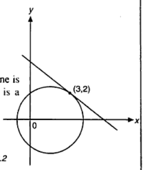


Fig.2.2

Example 4

A straight line through (0,–1) meets the curve $x^2 + y^2 - 4x - 2y + 4 = 0$ at the point (3,1). Find the coordinates of the second point where this line meets the curve.

First we find the equation of the straight line:

$$\frac{y+1}{1+1} = \frac{x-0}{3-0}$$

which gives $2x = 3y + 3$ or $x = \frac{3y+3}{2}$.

Then substitute for x in the equation of the curve:

$$\left(\frac{3y+3}{2}\right)^2 + y^2 - 4\left(\frac{3y+3}{2}\right) - 2y + 4 = 0$$

i.e. $\frac{9y^2 + 18y + 9}{4} + y^2 - 6y - 6 - 2y + 4 = 0.$

Clearing the fraction,

$$9y^2 + 18y + 9 + 4y^2 - 24y - 24 - 8y + 16 = 0$$

and so $13y^2 - 14y + 1 = 0$ or $(13y - 1)(y - 1) = 0$.

Hence $y = \frac{1}{13}$ or $y = 1$.

The corresponding values of x are then $\frac{21}{13}$ or 3.

So the second point is $\left(\frac{21}{13}, \frac{1}{13}\right)$.

Example 5

If the line $3x - 5y = 8$ meets the curve $\frac{3}{x} - \frac{1}{y} = 4$ at A and B, find the coordinates of the midpoint of AB.

First remove the fractions from the equation of the curve:

$$3y - x = 4xy \qquad \text{(i)}$$

From the linear equation, $x = \frac{8 + 5y}{3}$.

Substituting for x in (i),

$$3y - \frac{8 + 5y}{3} = 4y\left(\frac{8 + 5y}{3}\right)$$

Clearing the fraction, we have $9y - 8 - 5y = 4y(8 + 5y) = 32y + 20y^2$
or $20y^2 + 28y + 8 = 0$,
i.e. $5y^2 + 7y + 2 = 0$ or $(5y + 2)(y + 1) = 0$

Hence $y = -\frac{2}{5}$ or -1.
Then $x = 2$ or 1.
The coordinates of A and B are $(2, -\frac{2}{5})$ and $(1,-1)$ and the coordinates of the midpoint are therefore $(1\frac{1}{2}, -\frac{7}{10})$.

Example 6

If the sum of two numbers is 4 and the sum of their squares minus three times their product is 76, find the numbers.

Suppose the numbers are x and y.
The sum of the numbers is $x + y$.

Then $\quad x + y = 4 \qquad$ (i)

The (sum of the squares) – (3 × the product) is $x^2 + y^2 - 3xy$.

Then $\quad x^2 + y^2 - 3xy = 76 \qquad$ (ii)

We solve these equations.

From (i), $x = 4 - y$
Substituting in(ii):

$(4 - y)^2 + y^2 - 3y(4 - y) = 76$ which is then expanded.
$16 - 8y + y^2 + y^2 - 12y + 3y^2 = 76$
or $\quad 5y^2 - 20y - 60 = 0$

Hence $y^2 - 4y - 12 = 0$ which gives $(y - 6)(y + 2) = 0$ and $y = 6$ or -2.
Then from (i), the corresponding values of x are -2, and 6.
Therefore the two numbers are 6 and -2.
Arithmetically, there is only one solution. Geometrically, the line $x + y = 4$ meets the curve given by equation (ii) in two points $(6,-2)$ and $(-2,6)$.

Exercise 2.1 *(Answers on page 608.)*

1 Solve the following pairs of simultaneous equations:

(a) $x + y = 5,\ xy = x + 3$

(b) $x - y = 2,\ x(y + 2) = 9$

(c) $2x + y = 5,\ x^2 + y^2 = 10$

(d) $x - 2y = 2,\ x^2 + xy = 20$

(e) $2x + 3y = 5,\ y(y - x) = 5$

(f) $3x - 2y = 7,\ x^2 + y^2 = 10$

(g) $3x - y = 7,\ x^2 + xy - y^2 = 1$

(h) $x + 3y = 1,\ x^2 - xy + y^2 = 21$

(i) $3x + 4y = 2,\ x^2 - 3y^2 = 1$

(j) $3x + 2y = 13,\ 3x^2 + y^2 = 31$

(k) $\frac{x}{3} - \frac{y}{2} = 1,\ \frac{3}{x} + \frac{2}{y} = \frac{3}{2}$

(l) $\frac{x}{4} - \frac{y}{3} = 1,\ \frac{16}{x} + \frac{3}{y} = 3$

(m) $3x - 2y = 11,\ (x - 1)(y + 3) = 4$

2 The line $y = x + 2$ meets the curve $y^2 = 4(2x + 1)$ at A and B. Find the coordinates of the midpoint of AB.

3 Show that the line $x + y = 6$ is a tangent to the curve $x^2 + y^2 = 18$ and find the coordinates of the point of contact.

4 A line through (2,1) meets the curve $x^2 - 2x - y = 3$ at A(–2,5) and at B. Find the coordinates of B.

5 What is the relationship of the line $3x - 2y = 4$ to the curve $y = x - \frac{2}{x}$?

6 The perimeter of a rectangle is 22 cm and its area is 28 cm². Find its length and breadth.

7 The line through (1,6) perpendicular to the line $x + y = 5$ meets the curve $y = 2x + \frac{4}{x}$ again at P. Find the coordinates of P.

8 A(3,1) lies on the curve $(x - 1)(y + 1) = 4$. A line through A perpendicular to $x + 2y = 7$ meets the curve again at B. Find the coordinates of B.

9 The difference between two numbers is 2 and the difference of their squares is 28. Find the numbers.

10 Fencing is used to make 3 sides of a rectangle: two pieces each of length a m and one piece of length b m. The total length of fencing used is 30 m and the area enclosed is 100 m². What are the values of a and b?

11 The line $x - y = 7$ meets the curve $x^2 + y^2 - x = 21$ at A and B. Find the coordinates of the midpoint of AB.

12 The line through (–3,8) parallel to $y = 2x - 3$ meets the curve $(x + 3)(y - 2) = 8$ at A and B. Find the coordinates of the midpoint of AB.

SUMMARY

- To solve simultaneous equations, one linear, the other of the second degree:
  (a) make one of the variables the subject of the linear equation,
  (b) substitute in the second degree equation,
  (c) simplify and then solve the quadratic equation obtained,
  (d) find the corresponding values of the second variable.

  If two equal solutions are obtained, the line is a tangent to the curve given by the second degree equation.

REVISION EXERCISE 2 *(Answers on page 608.)*

A

1 Solve the simultaneous equations $4x - 3y = 11$ and $16x^2 - 3y^2 = 61$.

2 The line $y - 2x - 8 = 0$ meets the curve $y^2 + 8x = 0$ at A and B. Find the coordinates of the midpoint of AB. (C)

3 A straight line through the point (0,–3) intersects the curve $x^2 + y^2 - 27x + 41 = 0$ at (2,3). Calculate the coordinates of the point at which the line again meets the curve. (C)

4 Calculate the coordinates of the points of intersection of the straight line $2x + 3y = 10$ and the curve $\frac{2}{x} + \frac{3}{y} = 5$. (C)

5 Solve the simultaneous equations $2x + 3y = 6$ and $(2x + 1)^2 + 6(y - 2)^2 = 49$. (C)

6 The perimeter of the shape shown in Fig.2.3 is 90 cm and the area enclosed is 300 cm². All corners are right-angled. Find the values of x and y.

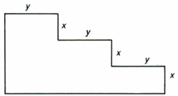


Fig.2.3

7 The point A(0,p) lies on the curve $y = (x - 2)^2$. A line through A perpendicular to $y = x + 3$ meets the curve again at B. Find
(a) the value of p,
(b) the coordinates of B.

8 Two quantities u and v are connected by the equation $u + 2v = 7$. A third quantity P, is given by $P = u(v - 3)$. Find the values of u and v when $P = -3$.

9 The hypotenuse of a right-angled triangle is $(2y - 1)$ cm long. The other two sides are x cm and $(y + 5)$ cm in length. If the perimeter of the triangle is 30 cm, find the possible values of x and y.

10 Solve the simultaneous equations $2x + 4y = 9$ and $4x^2 + 16y^2 = 20x + 4y - 19$. (C)

B

11 Solve the simultaneous equations $3x - 2y = 11$ and $x^2 + xy + y^2 = 7$.

12 A(3,4) and B(7,8) are two points. P(a,b) is equidistant from A and B such that AP = $\sqrt{26}$.

(a) Show that $a + b = 11$.

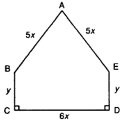
(b) Find the values of a and b.

13 In Fig.2.4, ABE is an isosceles triangle and BCDE is a rectangle. The total length round ABCDEA is 22 cm and the area enclosed is 30 cm^2.

(a) State the distance of A from BE in terms of x.

(b) Find the possible values of x and y.

Fig.2.4



14 Solve the simultaneous equations $x + y = 6$ and $\frac{1}{x-1} = \frac{3}{y} + \frac{1}{4}$.

15 The point P(a,b) lies on the line through A(–1,–2) and B(3,0) and PA = $\sqrt{125}$. Find the values of a and b.

16 A circle has centre (4,2) and radius $\sqrt{5}$ units. P(x,y) is any point on the circumference.

(a) Show that $x^2 + y^2 - 8x - 4y + 15 = 0$.

(b) Find the coordinates of the ends of the diameter which, when extended, passes through the origin.

(c) Find the coordinates of the ends of the perpendicular diameter.