Calculus (5) : e^x and $\ln x$

18

The function $\frac{1}{x}$ cannot be integrated by the usual rule. So if this function is to have an integral it cannot be an algebraic one but some other type of function. The story of its discovery is beyond our work but we can start with the origin of an important number in the story.

This comes from asking what happens to the value of $\left(1 + \frac{1}{t}\right)^t$ as $t \to \infty$ i.e. what is the value (if any) of $\lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^t$?

We shall not be able to *prove* what this limit is but the following set of values made by a calculator will suggest an answer.

t	$\left(1 + \frac{1}{t}\right)^t$
100	$(1.01)^{100} \approx 2.7048$
1000	$(1.001)^{1000} \approx 2.7169$
10 000	$(1.0001)^{10\,000} \approx 2.71815$
100 000	$(1.00001)^{100\,000} \approx 2.71827$
1 000 000	$(1.000001)^{1\,000\,000} \approx 2.71828$
10 000 000	$(1.0000001)^{10\,000\,000} \approx 2.71828$

As x increases, it appears that $\left(1 + \frac{1}{t}\right)^t$ tends to a value which is approximately 2.71828. This is true and we denote this limit by the letter **e**. Like π, e is an irrational number. Its importance is that it is taken as the base of **natural logarithms**, i.e. $\log_e x$ (written as $\ln x$.)

If $y = \log_e x = \ln x$ then $e^y = x$
If $y = e^x$ then $\ln y = x$

Similar to other logarithms, $\ln 1 = 0$, $\ln e = 1$ and if $0 < x < 1$, $\ln x$ is negative. If $x \le 0$, $\ln x$ is undefined.

So for example, if $y = e^{2x+3}$ then $\ln y = 2x + 3$; if $y = 2e^{3x}$ then $\ln y = \ln 2 + \ln e^{3x} = \ln 2 + 3x$.

We shall now see why such a strange number is chosen as a base for logarithms.

$\frac{d}{dx}$ ln x

Take $y = \ln x$ and let x have an increment δx. Consequently, y has an increment δy.

Then $y + \delta y = \ln(x + \delta x)$ and $\delta y = \ln(x + \delta x) - \ln x = \ln\left(\frac{x + \delta x}{x}\right)$.

Hence $\frac{\delta y}{\delta x} = \frac{1}{\delta x} \ln\left(\frac{x + \delta x}{x}\right) = \ln\left(1 + \frac{\delta x}{x}\right)^{\frac{1}{\delta x}}$.

To make use of the above limit, write $\frac{\delta x}{x} = \frac{1}{t}$ so $\frac{1}{\delta x} = \frac{t}{x}$.

Then $\frac{\delta y}{\delta x} = \ln\left(1 + \frac{1}{t}\right)^{\frac{t}{x}} = \frac{1}{x} \ln\left(1 + \frac{1}{t}\right)^{t}$.

Now let $\delta x \to 0$. Consequently, $\delta y \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$, $t \to \infty$ and $\left(1 + \frac{1}{t}\right)^t \to e$.

Using these, we have $\frac{dy}{dx} = \frac{1}{x} \ln e = \frac{1}{x}$ as $\ln e = \log_e e = 1$.

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

This is a very important and simple result. It is the basis of the work in this chapter and shows why e is taken as the base of natural logarithms.

Now using the rule for a composite function, we can differentiate $\ln f(x)$.

$\frac{d}{dx}$ ln f(x)

Suppose $y = \ln f(x)$.

Take $u = f(x)$ and so $y = \ln u$.

$\frac{dy}{du} = \frac{1}{u}$ and $\frac{du}{dx} = \frac{d}{dx} f(x) = f'(x)$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times f'(x) = \frac{f'(x)}{f(x)}$

$$\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$$

Example 1

Differentiate wrt x (a) $\ln(ax + b)$, (b) $\ln(x^2 - 3x + 1)$, (c) $\ln \sin 3x$, (d) $x^2 \ln x$, (e) $\frac{\ln x}{x + 1}$

(a) Here $f(x) = ax + b$.

$\frac{d}{dx} \ln(ax + b) = \frac{a}{ax + b}$ as $f'(x) = \frac{d}{dx}(ax + b) = a$.

(b) $\frac{d}{dx} \ln(x^2 - 3x + 1) = \frac{2x - 3}{x^2 - 3x + 1}$ (as $f'(x) = 2x - 3$)

(c) $\frac{d}{dx} \ln \sin 3x = \frac{3 \cos 3x}{\sin 3x} = 3 \cot 3x$

Exercise 18.1 *(Answers on page 643.)*

1 Differentiate the following wrt x and simplify where possible:

(a) $\ln 5x$ (b) $\ln x^2$ (c) $\ln(3x-1)$

(d) $\ln(\sin x)$ (e) $\ln(x+\tan x)$ (f) $\ln(\cos 2x)$

(g) $\ln(\cos^2 x)$ (h) $\ln\left(\sin \frac{x}{2}\right)$ (i) $\ln(2x^2-4x-1)$

(j) $\ln\sqrt{2x-5}$ (k) $\ln\left(\frac{1}{\sqrt{x}}\right)$ (l) $x\ln x$

(m) $\frac{\ln x}{x^2}$ (n) $\ln(x\cos x)$ (o) $\cos(\ln x)$

(p) $\ln x \ln 3x$ (q) $(x^2+1)\ln(x-1)$ (r) $(\ln x)^2$

(s) $\ln(\cos 3x)$ (t) $(x-1)\ln 2x$ (u) $\ln(x+\sin x)$

(v) $\ln(x+3)(2x-1)$ (w) $\ln\left(\frac{x-4}{x+3}\right)$

2 If $y = \ln(x+1)(x-2)$, show that $\frac{dy}{dx} = \frac{2x-1}{x^2-x-2}$.

3 If $y = \ln(3x+1)(2x-1)$, find and simplify $\frac{dy}{dx}$.

4 Given that $y = \ln\left(\frac{x-2}{x+1}\right)$, find $\frac{dy}{dx}$ in its simplest form.

5 Fig. 18.2 shows parts of two straight lines obtained by plotting $\ln y$ against x for two different functions. Each has two points marked. Find for each function, (a) y in terms of x, (b) the value of x when $y = 1$.

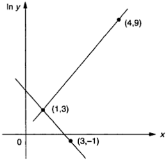

Fig. 18.2

6 State how the functions (a) $y = e^{3x-2}$ and (b) $y = 3e^{-2x}$ can each be represented by a straight line graph and give the equation of each line.

7 On graph paper, draw the graph of $y = e^x$ for $0 \le x \le 2$. By adding a suitable straight line, find an approximate solution to the equation $e^x + x = 5$.

8 Given that $e^{5x} = e^{3(y+1)}$ and that $\ln(3x+4y) = 2\ln 5$, form two simultaneous equations and hence find the value of x and of y.

9 Differentiate $\ln(x - \sin x)$. Hence find the gradient on the curve $y = \ln(x - \sin x)$ where $x = \pi$.

10 Differentiate $\frac{\sin x}{1 - \cos x}$. Hence show that $\frac{d}{dx}\ln\frac{\sin x}{1 - \cos x} = -\text{cosec } x$.

11 Differentiate $\frac{\cos x}{1 + \sin x}$. Hence show that $\frac{d}{dx}\ln\frac{\cos x}{1 + \sin x} = -\sec x$.

$\frac{d}{dx}e^x$

If $y = e^x$, then $\ln y = x$. Differentiating both sides wrt x, $\frac{1}{y}\frac{dy}{dx} = 1$ so $\frac{dy}{dx} = y = e^x$.

Hence $$\frac{d}{dx}e^x = e^x$$

This result makes e^x a unique function. It is the only function whose derivative is itself. The gradient at a point on the curve $y = e^x$ equals the value of y at that point. (This was suggested in Question 3 of Exercise 15.3).

$\frac{d}{dx}e^{f(x)}$

We can also differentiate composite functions of the type $e^{f(x)}$.

If $y = e^{f(x)}$ and $u = f(x)$, then $y = e^u$.

$\frac{dy}{du} = e^u$ and $\frac{du}{dx} = f'(x)$.

Hence $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u f'(x) = f'(x)e^{f(x)}$.

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

Example 4

Differentiate (a) e^{3x-2}, (b) $e^{\sin 2x}$, (c) xe^{-2x}, wrt x.

(a) $\frac{de^{3x-2}}{dx} = 3e^{3x-2}$ as $\frac{d(3x-2)}{dx} = 3$

(b) $\frac{d}{dx}e^{\sin 2x} = (2\cos 2x)\,e^{\sin 2x}$

(c) $y = xe^{-2x}$ is a product.

$\frac{dy}{dx} = e^{-2x} + x(-2e^{-2x}) = e^{-2x}(1 - 2x)$

Example 5

Find the coordinates of the point of intersection of the curves $y = e^{2x-1}$ and $y = e^{2-x}$ and the gradient of each curve at that point.

At the point of intersection, $e^{2x-1} = e^{2-x}$ so $2x - 1 = 2 - x$ and $x = 1$. The coordinates of the point are (1,e).

For $y = e^{2x-1}$, $\frac{dy}{dx} = 2e^{2x-1} = 2e$ when $x = 1$.

For $y = e^{2-x}$, $\frac{dy}{dx} = -e^{2-x} = -e$ when $x = 1$.

Exercise 18.2 *(Answers on page 643.)*

1 Differentiate wrt x:

(a) e^{4x}
(b) e^{5x-1}
(c) e^{5-3x}
(d) e^{x^2}
(e) $e^{\cos x}$
(f) xe^x
(g) $(2x-4)e^{-\frac{x}{2}}$
(h) e^{ax+b}
(i) e^{x^2+2x-1}
(j) $e^x \sin x$
(k) $\frac{e^x+1}{e^x}$
(l) $\frac{e^x}{x+1}$
(m) x^2e^{-x}
(n) $e^x - e^{-x}$
(o) $e^{-x}(\cos x - \sin x)$
(p) $(3x+2)e^{-2x}$
(q) $e^{2x} \ln x$
(r) $\frac{e^x}{e^x - 1}$
(s) $\frac{xe^x}{x-1}$
(t) $e^{2x} \cos 2x$
(u) $\frac{e^x}{x}$
(v) x^3e^{2x}
(w) $(e^x - e^{-x})^2$

2 Find the coordinates of the point where the curves $y = e^{5x-2}$ and $y = e^{4-x}$ meet and the gradient of each curve at that point.

3 Find the range of values of x for which $(x-3)e^{-2x}$ is increasing.

4 If $y = xe^{3x}$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence find the value of x for which y has a stationary point and state the nature of that point.

5 Given that $y = x^2e^{2x}$, find the values of x for which y is stationary.

6 If $y = (x^2 - 3)e^{-x}$, find the values of x where y is stationary and the nature of these points.

7 If $y = e^x \cos x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Use these to find the values of x $(0 < x < 2\pi)$ where y has stationary points and state the nature of these points.

8 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $y = e^x(\cos x + \sin x)$.
Hence find the values of x $(0 < x < 2\pi)$ where y is stationary and the nature of the stationary points.

9 Given that $y = e^x \sin x$, prove that $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$.

10 Find the gradient on the curve $y = e^{2x} \cos x$ where $x = 0$.

11 If $y = a^x$, show that $\ln y = x \ln a$ and hence find $\frac{dy}{dx}$.

12 If $\frac{d}{dx}\left(\frac{\sin x}{e^x}\right) = \frac{f(x)}{e^x}$, find f(x).

Integration of $\frac{1}{ax + b}$ and e^{ax+b}

We can now find an answer for $\int \frac{1}{x}\, dx$.

We know that $\frac{d}{dx} \ln x = \frac{1}{x}$ so $\int \frac{1}{x}\, dx = \ln x + c$.

However we must be careful. If $x < 0$, $\ln x$ is undefined. We can guard against this by writing

$$\int \frac{1}{x}\, dx = \ln |x| + c$$

This is justified as shown in Fig. 18.3 which shows the two branches of the curve $y = \frac{1}{x}$.

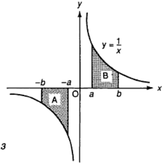

Fig. 18.3

The area $A = \int_{-b}^{-a} \frac{1}{x}\, dx = \left[\ln x\right]_{-b}^{-a}$ which is undefined.

By symmetry however, area A = area $B = \left[\ln x\right]_a^b = \left[\ln |x|\right]_{-a}^{-b}$.

Further, since $\frac{d}{dx} \ln(ax + b) = \frac{a}{ax + b}$,

then $$\int \frac{dx}{ax + b} = \frac{1}{a} \ln |ax + b| + c$$

As $\frac{d}{dx} e^{ax+b} = ae^{ax+b}$,

then $$\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b} + c$$

Note: These results only apply to the linear function $ax + b$.

Example 6

Find (a) $\int (3x-2)^{-1}\,dx$, *(b)* $\int_{-2}^{-1} \frac{dx}{2-3x}$, *(c)* $\int_{2}^{3} e^{2-x}\,dx$.

(a) $\int \frac{1}{3x-2}\,dx = \frac{1}{3}\ln|3x-2| + c$

(b) $\int_{-2}^{-1} \frac{dx}{2-3x} = \left[-\frac{1}{3}\ln|2-3x|\right]_{-2}^{-1}$

$= \left(-\frac{1}{3}\ln|5|\right) - \left(-\frac{1}{3}\ln|8|\right)$

$= -\frac{1}{3}\ln 5 + \frac{1}{3}\ln 8 = \frac{1}{3}\ln\frac{8}{5} \approx 0.16$

Such results however are usually left in terms of ln.

(c) $\int_{2}^{3} e^{2-x}\,dx = \left[-e^{2-x}\right]_{2}^{3}$

$= (-e^{-1}) - (-e^{0}) = -e^{-1} + 1 = 1 - \frac{1}{e}$.

Example 7

Find the area of the region enclosed by the curve $y = \frac{1}{x}$, *the x-axis and the lines* $x = 1, x = 3$.

Area $= \int_{1}^{3} \frac{1}{x}\,dx = \left[\ln|x|\right]_{1}^{3} = \ln 3 - \ln 1 = \ln 3$ units^2

Example 8

The part of the curve $y = \frac{1}{\sqrt{2-x}}$ *between* $x = -3$ *and* $x = -1$ *is rotated about the x-axis through 360°. Find the volume of the solid created.*

Volume $= \int_{-3}^{-1} \pi y^2\,dx = \pi\int_{-3}^{-1} \frac{1}{2-x}\,dx$

$= \pi\left[-\ln|2-x|\right]_{-3}^{-1} = \pi(-\ln|3|) - \pi(-\ln|5|) = \pi\ln\frac{5}{3}$ units^3

Example 9

The region enclosed by the curves $y = e^x$ *and* $y = e^{2x}$ *and the lines* $x = 1, x = 2$, *is rotated about the x-axis through 360°. Find, in terms of e, the volume of the solid formed.*

Volume $= \pi\int_{1}^{2} [(e^{2x})^2 - (e^x)^2]\,dx$

$$= \pi \int_1^2 (e^{4x} - e^{2x})\, dx = \pi \left[\frac{1}{4} e^{4x} - \frac{1}{2} e^{2x} \right]_1^2$$

$$= \pi\left(\frac{e^8}{4} - \frac{e^4}{2}\right) - \pi\left(\frac{e^4}{4} - \frac{e^2}{2}\right) = \frac{\pi}{4}(e^8 - 3e^4 + 2e^2)$$

$$= \frac{\pi e^2}{4}(e^6 - 3e^2 + 2) \text{ units}^3$$

Exercise 18.3 *(Answers on page 643.)*

1 Find

(a) $\int e^{3x}\, dx$ (b) $\int e^{-x}\, dx$ (c) $\int e^{2-x}\, dx$

(d) $\int \frac{dx}{2x + 3}$ (e) $\int \frac{dx}{3 - 2x}$ (f) $\int (4 - x)^{-1}\, dx$

(g) $\int \frac{x + 1}{x}\, dx$ (h) $\int \left(e^x + \frac{1}{e^x}\right)^2 dx$ (i) $\int e^{-3x}\, dx$

(j) $\int e^{1-2x}\, dx$ (k) $\int \frac{dx}{3x + 1}$ (l) $\int \frac{x + 2}{x^2}\, dx$

2 Evaluate the following, giving the result in terms of e:

(a) $\int_0^1 e^{4x}\, dx$ (b) $\int_{-1}^0 e^{-x}\, dx$ (c) $\int_{-3}^{-1} e^{1-x}\, dx$

(d) $\int_1^3 e^{2x-1}\, dx$ (e) $\int_{-2}^0 \frac{2}{e^{2x}}\, dx$ (f) $\int_1^2 e^{-2x}\, dx$

(g) $\int_{-2}^0 e^{-\frac{x}{2}}\, dx$ (h) $\int_{-1}^1 e^{2-x}\, dx$

3 Express the following in terms of ln:

(a) $\int_1^2 \frac{dx}{x - 3}$ (b) $\int_{-2}^{-1} \frac{dx}{3 - x}$ (c) $\int_0^2 \frac{dx}{2x + 3}$

(d) $\int_0^1 (2 - x)^{-1}\, dx$ (e) $\int_{-\frac{4}{3}}^{-\frac{1}{3}} \frac{dx}{2 - 3x}$ (f) $\int_{\frac{1}{2}}^2 \frac{3x + 1}{2x}\, dx$

(g) $\int_{-2}^0 (5 - x)^{-1}\, dx$ (h) $\int_1^2 \frac{dx}{3 - 4x}$ (i) $\int_{-4}^0 \frac{dx}{1 - 2x}$

4 Show that $\frac{1}{x - 1} - \frac{1}{x + 1} = \frac{2}{x^2 - 1}$. Hence evaluate $\int_2^3 \frac{dx}{x^2 - 1}$.

5 Show that $\frac{1}{x - 4} - \frac{1}{x - 2} = \frac{2}{x^2 - 6x + 8}$ and hence evaluate $\int_5^6 \frac{dx}{x^2 - 6x + 8}$.

6 P is a function of t such that $\frac{dP}{dt} = e^{-3t}$ and $P = 3$ when $t = 0$. Find P in terms of t.

7 Calculate the area of the region enclosed by the curves $y = e^x$ and $y = e^{2x}$ and the line $x = 1$.

8 Find $\int \frac{e^x + 1}{e^x}\, dx$.

9 The part of the curve $y = \frac{1}{\sqrt{x + 2}}$ between $x = 1$ and $x = 3$ is rotated about the x-axis through 360°. Find the volume formed.

10 (a) The gradient of a curve is given by $\frac{dy}{dx} = e^{2x}$ and it passes through the point $\left(0, \frac{1}{2}\right)$. Find the equation of the curve.

(b) Find the area of the region enclosed by the curve, the x-axis and the lines $x = 0$ and $x = 1$.

11 (a) Sketch the curves $y = e^x$ and $y = \ln x$.

(b) Differentiate $x \ln x - x$ wrt x and hence find $\int \ln x \, dx$.

(c) Hence find the area of the region enclosed by these curves, the x-axis, the y-axis and the line $x = 2$.

12 (a) Find the equation of the tangent to the curve $y = \ln x$ at the point where $x = 1$.

(b) Find the area between this tangent, the curve and the line $x = 2$. (Use the result of Question 11(b).)

SUMMARY

- $\ln x = \log_e x$; $e \approx 2.7183$
- If $y = \ln x$, $x = e^y$; if $y = e^{ax+b}$, $\ln y = ax + b$
- $\frac{d}{dx} \ln x = \frac{1}{x}$ $\qquad$ $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$ $\qquad$ $\frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}$
- $\int \frac{dx}{x} = \ln |x| + c$ $\qquad$ $\int \frac{dx}{ax + b} = \frac{1}{a} \ln |ax + b| + c$
- $\int e^x \, dx = e^x + c$ $\qquad$ $\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$

Note: $\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$ for all values of n *except* $n = -1$.

REVISION EXERCISE 18 *(Answers on page 643.)*

A

1 Evaluate

(a) $\int_0^1 e^{1-2x} \, dx$ $\qquad$ (b) $\int_1^2 \frac{dx}{x - 5}$ $\qquad$ (c) $\int_{-2}^0 e^{1-\frac{x}{2}} \, dx$

(d) $\int_a^b e^{kt} \, dt$ (k is a constant) $\qquad$ (e) $\int_{-3}^{-1} e^{2-x} \, dx$ $\qquad$ (f) $\int_{-2}^{-1} \frac{dx}{4 - x}$

(g) $\int_0^1 \frac{e}{e^{2x}} \, dx$

2 Find the gradient of the curve $y = \ln(x + \sin 2x)$ where $x = \frac{\pi}{2}$.

3 Sketch the curve $y = e^{-2x}$ for $x > 0$ and find the area of the region enclosed by the curve, the y-axis, the x-axis and the line $x = 1$.