Calculus (4): Further Techniques: Trigonometric Functions

17

So far we can differentiate single terms such as x^5, polynomials such as $2x^3 - 3x + \frac{1}{x}$ and composite functions such as $(2x^3 - 1)^4$.
We now extend the range of functions we can deal with.

Fractional Indices

If $y = ax^n$, then you will recall that $\frac{dy}{dx} = nax^{n-1}$ and $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$.

The rules for differentiation and integration still hold when the index n, is a rational number, i.e. a fraction.

Example 1

Differentiate (a) $x^{\frac{2}{3}}$, (b) $\frac{1}{\sqrt{x}}$, (c) $\sqrt{x^2 - 2x - 3}$ wrt x.

(a) $y = x^{\frac{2}{3}}$

Then $\frac{dy}{dx} = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}}$

(b) $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

(c) $y = (x^2 - 2x - 3)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(x^2 - 2x - 3)^{-\frac{1}{2}} \times (2x - 2)$

$= (x - 1)(x^2 - 2x - 3)^{-\frac{1}{2}}$ or $\frac{x-1}{\sqrt{x^2 - 2x - 3}}$

Example 2

Find (a) $\int x^{-\frac{1}{3}}\,dx$, *(b)* $\int_4^9 2x^{\frac{3}{2}}\,dx$.

(a) $\int x^{-\frac{1}{3}}\,dx = \dfrac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + c = \dfrac{x^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3}{2}x^{\frac{2}{3}} + c$

Check by differentiating.

(b) $\int_4^9 2x^{\frac{3}{2}}\,dx = \left[\dfrac{2x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_4^9$

$= \left[\dfrac{2x^{\frac{5}{2}}}{\frac{5}{2}}\right]_4^9 = \left[\frac{4}{5}x^{\frac{5}{2}}\right]_4^9$

$= \left(\frac{4}{5} \times 9^{\frac{5}{2}}\right) - \left(\frac{4}{5} \times 4^{\frac{5}{2}}\right)$

$= \left(\frac{4}{5} \times 243\right) - \left(\frac{4}{5} \times 32\right) = \frac{4}{5}(243 - 32) = \frac{4}{5} \times 211 = 168\frac{4}{5}$

Integration of Powers of the Linear Function $ax + b$

If $y = (ax + b)^{n+1}$, then $\frac{dy}{dx} = (n + 1)a(ax + b)^n$.

Hence $\int (n + 1)a(ax + b)^n\,dx = (ax + b)^{n+1}$

and so $$\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{(n + 1)a} + c$$ where $n \neq -1$

This result only applies to a **linear function** $ax + b$. The integration of powers of non-linear functions such as $ax^2 + b$ cannot be done in this way and is outside our work.

The case where $n = -1$ will be studied in Chapter 18.

Example 3

Find (a) $\int (2x - 1)^3\,dx$, *(b)* $\int \dfrac{dx}{(3x + 2)^2}$ *(c)* $\int_1^5 \sqrt{2x - 1}\,dx$.

(a) Here $a = 2$, $b = -1$ and $n = 3$.

So $\int (2x - 1)^3\,dx = \dfrac{(2x - 1)^4}{4 \times 2} + c = \frac{1}{8}(2x - 1)^4 + c$

(b) $\int \dfrac{dx}{(3x + 2)^2}$ is short for $\int \dfrac{1}{(3x + 2)^2}\,dx = \dfrac{(3x + 2)^{-1}}{(-1)(3)} + c$

$= -\frac{1}{3}(3x + 2)^{-1} + c$

$= -\dfrac{1}{3(3x + 2)} + c$

(c) $\int_1^5 \sqrt{2x-1}\,dx = \int_1^5 (2x-1)^{\frac{1}{2}}\,dx$

$$= \left[\frac{(2x-1)^{\frac{3}{2}}}{2 \times \frac{3}{2}}\right]_1^5$$

$$= \left(\tfrac{1}{3} \times 9^{\frac{3}{2}}\right) - \left(\tfrac{1}{3} \times 1^{\frac{3}{2}}\right) = 9 - \tfrac{1}{3} = 8\tfrac{2}{3}$$

Example 4

Find the area bounded by the curve $y = \frac{1}{\sqrt{2x-3}}$, the x-axis and the lines x = 2, x = 6.

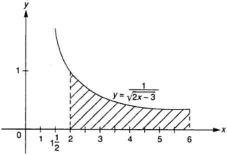

Fig.17.1

As $\sqrt{2x-3}$ is positive for $x > 1\frac{1}{2}$, y is real and positive in the area required.

$$\text{Area} = \int_2^6 (2x-3)^{-\frac{1}{2}}\,dx = \left[\frac{(2x-3)^{\frac{1}{2}}}{2 \times \frac{1}{2}}\right]_2^6$$

$$= \left[(2x-3)^{\frac{1}{2}}\right]_2^6 = \left(9^{\frac{1}{2}}\right) - \left(1^{\frac{1}{2}}\right) = 2$$

Example 5

The section of the curve $y = \frac{1}{\sqrt[3]{x-1}}$ between the lines x = 2 and x = 9 is rotated about the x-axis through 360°. Find the volume of the solid created.

You will recall that the volume of a solid of revolution $= \int \pi y^2\,dx$.

Here $y = (x-1)^{-\frac{1}{3}}$.

So the volume $= \int_2^9 \pi(x-1)^{-\frac{2}{3}}\,dx$

$$= \left[\frac{\pi(x-1)^{\frac{1}{3}}}{\frac{1}{3}}\right]_2^9 = \left[3\pi(x-1)^{\frac{1}{3}}\right]_2^9 = (3\pi \times 2) - (3\pi \times 1) = 3\pi$$

Example 6

If $y = 2\sqrt{9 - x^2}$, what is the approximate change in y when x is increased from 2 to 2.01?

When $x = k$, $\delta y \approx \left(\frac{dy}{dx}\right)_{x=k} \delta x$, where $\left(\frac{dy}{dx}\right)_{x=k}$ is the value of $\frac{dy}{dx}$ when $x = k$.

Here $k = 2$ and $\delta x = 0.01$.

$y = 2(9 - x^2)^{\frac{1}{2}}$ so $\frac{dy}{dx} = 2 \times \frac{1}{2}(9 - x^2)^{-\frac{1}{2}}(-2x) = -2x(9 - x^2)^{-\frac{1}{2}}$.

$\left(\frac{dy}{dx}\right)_{x=2} = -4(5)^{-\frac{1}{2}} = \frac{-4}{\sqrt{5}}$

Hence $\delta y \approx -\frac{4}{\sqrt{5}} \times 0.01 = -0.018$ (y has decreased).

Exercise 17.1 *(Answers on page 640.)*

1 Differentiate wrt x:

(a) $x^{-\frac{3}{4}}$ (b) $4\sqrt{x}$ (c) $\sqrt{4x - 3}$

(d) $2x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$ (e) $\sqrt[3]{x^3 - 6x^2}$ (f) $2x^{\frac{3}{2}}$

(g) $\frac{4}{\sqrt{x}}$ (h) $\sqrt{4x^3 - 3}$ (i) $\sqrt{1 - 2x + 4x^2}$

2 Integrate wrt x:

(a) $x^{\frac{3}{4}}$ (b) $x^{-\frac{1}{2}}$ (c) $x^{\frac{2}{3}} - x^{-\frac{1}{3}}$

(d) $\frac{1}{2}x^{-\frac{2}{3}}$ (e) $\frac{x^{\frac{2}{3}} - x^{\frac{1}{2}}}{x^{\frac{1}{3}}}$ (f) $5x^{\frac{2}{3}}$

(g) $\frac{3}{\sqrt{x}}$ (h) $\frac{1}{3}x^{-\frac{1}{3}}$ (i) $\frac{x - x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$

3 Evaluate

(a) $\int_1^4 \frac{\sqrt{x}}{2}\,dx$ (b) $\int_{-8}^{-1} x^{\frac{1}{3}}\,dx$ (c) $\int_0^1 x^{\frac{1}{5}}\,dx$ (d) $\int_1^4 x^{-\frac{2}{3}}\,dx$ (e) $\int_1^{16} x^{-\frac{1}{2}}\,dx$

4 Integrate wrt x:

(a) $(2x - 3)^2$ (b) $(2x + 5)^4$ (c) $(x - 2)^{-3}$

(d) $\sqrt{x - 3}$ (e) $\frac{1}{(3x - 2)^2}$ (f) $(2x + 3)^{-\frac{1}{2}}$

(g) $\frac{1}{\sqrt{2x + 3}}$ (h) $(3 - 4x)^3$ (i) $\frac{1}{\sqrt{3 - 2x}}$

(j) $(3x + 2)^4$ (k) $(4x - 1)^{\frac{1}{3}}$ (l) $(2x - 5)^{-\frac{1}{2}}$

(m) $\sqrt[3]{4x - 1}$ (n) $(1 - 2x)^{-2}$

5 Find the values of

(a) $\int_0^1 (2x + 1)^2\,dx$ (b) $\int_1^5 \sqrt{3x + 1}\,dx$

(c) $\int_0^1 (3x - 1)^2\,dx$ (d) $\int_{-5}^0 \sqrt{1 - 3x}\,dx$

(e) $\int_{\frac{2}{3}}^{\frac{4}{3}} (3x - 4)^3\,dx$ (f) $\int_{-2}^2 \sqrt{2x + 5}\,dx$

(g) $\int_1^6 \frac{dx}{\sqrt{x+3}}$ (h) $\int_1^2 (3x-2)^3\,dx$

6 Calculate the area bounded by the curve $y = (3x-1)^{-2}$, the x-axis and the lines $x = 1$, $x = 3$.

7 The part of the curve $y = \frac{1}{2x-3}$ between $x = 2$ and $x = 3$ is rotated about the x-axis through 360°. Find the volume of the solid of revolution.

8 If $y = 3\sqrt{x}$, find the approximate change in y when x is increased from 4 to 4.01.

9 Given that $y = 3\sqrt{9+x^2}$, find the change in y approximately when x is decreased from 4 to 3.99.

10 Given that $T = 9r^{\frac{4}{3}}$ and that r is increased from 8 to 8.01, find the approximate change in T.

11 If $P = kv^{\frac{2}{3}}$, where k is a constant, find the approximate percentage change in P if v is increased by 3% when it is 5.

12 If $V = 10x^{\frac{3}{2}}$, find the approximate change in V when x is decreased from 4 to 3.998.

Differentiation of the Product of Two Functions

$y = (3x-1)^3(x^2+5)^2$ is a product of two functions of x, $(3x-1)^3$ and $(x^2+5)^2$. Each of these can be differentiated but how can we find $\frac{dy}{dx}$? As we shall see, the result is NOT the product of their derivatives.

Let $y = uv$ where u and v are each functions of x.

Suppose x has an increment δx. This will produce increments δu in u and δv in v and finally produce an increment δy in y.

So $y + \delta y = (u+\delta u)(v+\delta v) = uv + u\delta v + v\delta u + (\delta u)(\delta v)$

Then $\delta y = u\delta v + v\delta u + (\delta u)(\delta v)$

and $\frac{\delta y}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\delta v$

Now let $\delta x \to 0$. Consequently $\delta u \to 0$, $\delta v \to 0$, $\frac{\delta u}{\delta x} \to \frac{du}{dx}$, $\frac{\delta v}{\delta x} \to \frac{dv}{dx}$ and $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$.

So, as $\delta x \to 0$, $\frac{dy}{dx} \to u\frac{dv}{dx} + v\frac{du}{dx}$.

Hence we have the **product rule** for $y = uv$:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

where u and v are functions of x.

As the result is symmetrical in u and v, it does not matter which function is chosen as u or v.

Example 7

Differentiate $(3x-2)(x^3+4)$ wrt x.

Take $u = 3x - 2$, $v = x^3 + 4$.

$\frac{du}{dx} = 3$, $\frac{dv}{dx} = 3x^2$

Then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (3x - 2) \times 3x^2 + (x^3 + 4) \times 3$

$= 9x^3 - 6x^2 + 3x^3 + 12 = 12x^3 - 6x^2 + 12$

Example 8

Differentiate $x^3(2x - 1)^4$ wrt x.

Take $u = x^3$, $v = (2x - 1)^4$.

$$\frac{dy}{dx} = \underset{\uparrow \atop u}{x^3} \times \underset{\uparrow \atop \frac{dv}{dx}}{\underbrace{4(2x-1)^3 \times 2}} + \underset{\uparrow \atop v}{(2x-1)^4} \times \underset{\uparrow \atop \frac{du}{dx}}{3x^2}$$

In this example, we simplify as far as possible and leave the result in factor form.

$\frac{dy}{dx} = x^2(2x - 1)^3[x \times 8 + (2x - 1)3]$

$= x^2(2x - 1)^3(14x - 3)$

Example 9

Differentiate $(3x - 1)^3(x^2 + 5)^2$ wrt x.

$\frac{dy}{dx} = (3x - 1)^3 \times 2(x^2 + 5) \times 2x + (x^2 + 5)^2 \times 3(3x - 1)^2 \times 3$

$= (3x - 1)^2(x^2 + 5)[(3x - 1) \times 4x + (x^2 + 5) \times 9]$

$= (3x - 1)^2(x^2 + 5)(12x^2 - 4x + 9x^2 + 45)$

$= (3x - 1)^2(x^2 + 5)(21x^2 - 4x + 45)$

Exercise 17.2 *(Answers on page 641.)*

1 Differentiate each of the following products wrt x. Leave the answers in simplified factor form.

(a) $x(x - 2)^2$ (b) $x^2(x^2 - 1)$ (c) $(x^2 + 1)(x^3 - 1)$
(d) $(x + 1)^2(x - 2)^3$ (e) $x^5(1 - 2x)^2$ (f) $(1 - x)^2(3 - x)^3$
(g) $x^2(x^2 - x - 1)^3$ (h) $x^2(x^2 - 3)^3$ (i) $(3x - 2)^2(2x^2 - 1)$
(j) $(x^2 + 1)^2(2x - 1)^3$ (k) $\sqrt{x}(x^3 - 1)^2$ (l) $x(\sqrt{x} - 1)^2$
(m) $2x(1 - 2x)^3$ (n) $\sqrt{x - 1}(x + 1)^4$ (o) $(x^2 - x - 2)(x + 1)^3$
(p) $(3x - 1)^2(2x + 3)^3$

2 Find the equation of the tangent to the curve $y = (x + 1)(x - 2)^3$ at the point where $x = 1$.

(b) $u = x^2 + 1,\ v = x^2 - x - 1,\ \frac{du}{dx} = 2x,\ \frac{dv}{dx} = 2x - 1$

$$\frac{dy}{dx} = \frac{(x^2 - x - 1)(2x) - (x^2 + 1)(2x - 1)}{(x^2 - x - 1)^2}$$

$$= \frac{2x^3 - 2x^2 - 2x - 2x^3 - 2x + x^2 + 1}{(x^2 - x - 1)^2} = \frac{-x^2 - 4x + 1}{(x^2 - x - 1)^2}$$

(c) $u = x,\ v = (x + 1)^{\frac{1}{2}},\ \frac{du}{dx} = 1,\ \frac{dv}{dx} = \frac{1}{2}(x + 1)^{-\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{(x + 1)^{\frac{1}{2}}(1) - x\frac{1}{2}(x + 1)^{-\frac{1}{2}}}{[(x + 1)^{\frac{1}{2}}]^2} = \frac{(x + 1)^{\frac{1}{2}} - \frac{x}{2}(x + 1)^{-\frac{1}{2}}}{x + 1}$$

To simplify this, multiply the numerator and denominator by $2(x + 1)^{\frac{1}{2}}$.

$$\frac{dy}{dx} = \frac{2(x + 1) - x}{2(x + 1)(x + 1)^{\frac{1}{2}}} = \frac{x + 2}{2(x + 1)^{\frac{3}{2}}}$$

Example 11

If $y = \frac{x}{3x + 2}$, *show that* $\frac{dy}{dx} = \frac{2}{(3x + 2)^2}$.

Hence or otherwise find $\int_1^3 \frac{dx}{(3x + 2)^2}$.

$$\frac{dy}{dx} = \frac{(3x + 2)(1) - x(3)}{(3x + 2)^2} = \frac{2}{(3x + 2)^2}$$

Hence means that we should use the above result and notice that

$$\int_1^3 \frac{dx}{(3x + 2)^2} = \frac{1}{2}\int_1^3 \frac{2}{(3x + 2)^2}\,dx$$

$$= \frac{1}{2}\left[\frac{x}{3x + 2}\right]_1^3 \quad \text{(i)}$$

$$= \frac{1}{2}\left(\frac{3}{11}\right) - \frac{1}{2}\left(\frac{1}{5}\right) = \frac{2}{55}$$

Otherwise means that another method can be used. We must notice that it is the integral of a linear function.

$$\int_1^3 (3x + 2)^{-2}\,dx = \left[\frac{(3x + 2)^{-1}}{-3}\right]_1^3$$

$$= \left[\frac{-1}{3(3x + 2)}\right]_1^3 \quad \text{(ii)}$$

$$= \left(-\frac{1}{33}\right) - \left(-\frac{1}{15}\right) = \frac{2}{55}$$

Note: The two integrals (i) and (ii) look different but they only differ by a constant.

$$\frac{1}{2}\left(\frac{x}{3x + 2}\right) = \frac{1}{6}\left(\frac{3x + 2 - 2}{3x + 2}\right) = \frac{1}{6}\left(1 - \frac{2}{3x + 2}\right) = \frac{1}{6} - \frac{-1}{3(3x + 2)}.$$

The constant $\frac{1}{6}$ disappears when the limits are substituted.

Exercise 17.3 *(Answers on page 641.)*

1 Differentiate wrt x, simplifying where possible:

(a) $\frac{x}{x+2}$ (b) $\frac{x+1}{x+2}$ (c) $\frac{x-2}{2x+1}$

(d) $\frac{3x-2}{x^2+1}$ (e) $\frac{x^2+2}{x-1}$ (f) $\frac{x^2+x-1}{1-x}$

(g) $\frac{x}{\sqrt{x-2}}$ (h) $\frac{2x}{\sqrt{2x+1}}$ (i) $\frac{x^3}{x^2+1}$

(j) $\frac{x}{\sqrt{x^2-2}}$ (k) $\frac{x^2}{3x+1}$ (l) $\frac{x-1}{2-x}$

(m) $\frac{3x-4}{x^2+1}$ (n) $\frac{x+1}{x^2+1}$

2 If $y = \frac{x}{x+1}$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence show that $(1+x)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$.

3 If $y = \frac{x}{2x-1}$, show that $\frac{dy}{dx} = \frac{-1}{(2x-1)^2}$. Hence or otherwise find $\int_1^5 \frac{dx}{(2x-1)^2}$.

4 Given that $y = \frac{x}{2x+3}$, find $\frac{dy}{dx}$.

Hence or otherwise evaluate $\int_1^2 \frac{dx}{(2x+3)^2}$.

5 If $y = \sqrt{\frac{x}{x+1}}$, find $\frac{dy}{dx}$. (Take $y = \frac{\sqrt{x}}{\sqrt{x+1}}$.)

Hence evaluate $\int_{-\frac{1}{3}}^{-\frac{9}{8}} \frac{dx}{x^{\frac{1}{2}}(x+1)^{\frac{3}{2}}}$.

6 Differentiate $\frac{x}{\sqrt{x^2+3}}$ wrt x. Hence find $\int_{-1}^{1} \frac{dx}{(x^2+3)^{\frac{3}{2}}}$.

7 Find $\frac{dy}{dx}$ if $y = \sqrt{\frac{x+1}{x-1}}$.

8 Find the values of x which give stationary points on the curve $y = \frac{x^2}{x-2}$.

9 (a) Given that $y = \frac{x+a}{x+2}$ and that $\frac{dy}{dx} = -\frac{1}{25}$ when $x = 3$, find the value of a.

(b) If $\int_0^1 \frac{1}{(2x+k)^2}\,dx = \frac{1}{3}$, where k is a constant, find the value of k.

10 If $y = \frac{x}{\sqrt{x^2+x+1}}$, find $\frac{dy}{dx}$.

Hence find the x-coordinate of the stationary point on the curve.

Differentiation of Implicit Functions

All the functions we have met so far have been in the form $y = f(x)$ i.e. they have been **explicit** functions. y has been given directly in terms of x. A function may however be stated **implicitly**, as for example $x^3 + y^3 = 3xy$, where it would be difficult to make y the subject. Using the product rule we can differentiate such functions and *then* find $\frac{dy}{dx}$.

Using a calculator, the following values of sin x and x were obtained:

x (radians)	$\sin x$
0.2	0.198669
0.1	0.09983
0.05	0.049979
0.01	0.0099998
0.001	0.0009999

This shows that when x is small, $\sin x \approx x$. It would suggest that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. Here is a simple proof of this.

In Fig. 17.2, OAB is a sector of a circle centre O, radius r and angle x radians. AC is perpendicular to OA. Then AC = $r \tan x$.

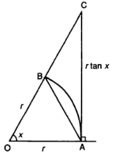

Fig. 17.2

Area of △AOB < area of sector AOB < area of △AOC,

i.e. $\frac{1}{2}r^2 \sin x \quad < \quad \frac{1}{2}r^2 x \quad < \quad \frac{1}{2}r^2 \tan x$

Hence $\sin x < x < \tan x$.

Dividing by $\sin x$, $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$.

Now as $x \to 0$, $\cos x \to 1$ and $\frac{1}{\cos x} \to 1$.

The left hand term is fixed at 1 and the right hand term $\to 1$. Hence the middle term must $\to 1$. Therefore $\lim_{x \to 0} \frac{x}{\sin x} = 1$.

In a more convenient form,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Note: For this result to be valid, x must be in radians.

Example 15

Differentiate (a) sin 3x, (b) sin(ax + b), (c) sin² x, (d) sin³(3x – 2).

(a) $y = \sin 3x$

We treat this as a composite function, i.e. $y = \sin u$ where $u = 3x$.

$\frac{dy}{du} = \cos u$ and $\frac{du}{dx} = 3$.

Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times 3 = 3 \cos 3x$.

Note that the function **sin** is differentiated first to give cos, then the angle $\mathbf{3x}$ is differentiated to give 3.

(b) $y = \sin(ax + b)$

Taking $y = \sin u$ where $u = ax + b$,

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \cos u \times a = a \cos(ax + b)$

Note this result for future use:

$\frac{d}{dx} \sin(ax + b) = a \cos(ax + b)$

First differentiate the function, then the angle.

(c) $y = \sin^2 x$

Treat this as a power of the function sin x.

Take $y = u^2$ where $u = \sin u$.

First differentiate as a power, i.e. $\frac{dy}{du}$, then differentiate the function sin, i.e. $\frac{du}{dx}$.

$\frac{dy}{dx} =$	$2 \sin x$	$\times$	$\cos x$	$= 2 \sin x \cos x$
	differentiate $\sin^2 x$ to get $2 \sin x$		differentiate $\sin x$ to get $\cos x$	

(The result could also be written as $\sin 2x$).

(d) $y = \sin^3 (3x - 2)$

First differentiate as a **power,** then differentiate **sin**, then the angle.

$\frac{dy}{dx} =$	$3 \sin^2(3x - 2)$	$\times$	$\cos(3x - 2)$	$\times$	3
	differentiate $\sin^3$ to get $3 \sin^2$		differentiate sin to get cos		differentiate $3x - 2$ to get 3

The sequence is $\sin^3(3x - 2)$ (power first; function second; variable last)

Hence $\frac{dy}{dx} = 9 \sin^2(3x - 2) \cos(3x - 2)$

Example 16

Differentiate sin x° wrt x.

We must first convert the angle to radians.

$x^\circ = \frac{\pi x}{180}$ radians

If $y = \sin \frac{\pi x}{180}$, then

$\frac{dy}{dx} = \cos \frac{\pi x}{180} \times \frac{\pi}{180}$ or $\frac{\pi}{180} \cos x^\circ$.

Note that the result is NOT cos x°. All formulae in calculus for trigonometrical functions are only true for radian measure. Angles in degrees must be converted to radians.

Example 17

Differentiate (a) x sin x, (b) $\sqrt{1 - \sin x}$ wrt x.

(a) This is a product of x and sin x.

If $y = x \sin x$, then $\frac{dy}{dx} = x \cos x + \sin x$.

(b) If $y = (1 - \sin x)^{\frac{1}{2}}$, $\frac{dy}{dx} = \frac{1}{2}(1 - \sin x)^{-\frac{1}{2}} \times (-\cos x)$

$= \frac{-\cos x}{2\sqrt{1 - \sin x}}$

Example 18

Find the values of x for $0 < x < \pi$ which satisfy the equation $\frac{d}{dx}(x - \sin 2x) = \sin^2 x$.

$\frac{d}{dx}(x - \sin 2x) = 1 - 2\cos 2x = 1 - 2(1 - 2\sin^2 x) = 4\sin^2 x - 1$

Hence $4\sin^2 x - 1 = \sin^2 x$ or $\sin x = \pm \frac{1}{\sqrt{3}}$.

Solving this equation, $x = 0.62$ or 2.52 radians ($x < \pi = 3.14$).

Example 23

Differentiate wrt x (a) x tan 2x, (b) sin x tan x.

(a) If $y = x \tan 2x$, $\frac{dy}{dx} = x \sec^2 2x \times 2 + \tan 2x$
$= 2x \sec^2 2x + \tan 2x$.

(b) If $y = \sin x \tan x$, $\frac{dy}{dx} = \sin x \sec^2 x + \tan x \cos x$
$= \sin x \sec^2 x + \sin x$
$= (\sin x)(\sec^2 x + 1)$

Exercise 17.6 *(Answers on page 642.)*

1 Differentiate wrt x:

(a) $\sin 3x$ (b) $\sin \frac{x}{2}$ (c) $\cos \frac{x}{4}$
(d) $\tan 3x$ (e) $\operatorname{cosec} x$ (f) $x \sin x$
(g) $x^2 \sin 2x$ (h) $\cos(2x^2 - 1)$ (i) $\sin\left(\frac{\pi}{3} - x\right)$
(j) $\tan \frac{x}{2}$ (k) $x \sin x + \cos x$ (l) $\frac{\cos x}{2 - \sin x}$
(m) $\cos^3 2x$ (n) $x \cos x - \sin 2x$ (o) $\sin 3x \cos 2x$
(p) $\sqrt{4 + \sin^2 2x}$ (q) $\cos^3(1 - 3x^2)$ (r) $\sqrt{\tan 2x}$

2 Differentiate wrt x:

(a) $\cos 3x$ (b) $\sin \frac{x}{3}$ (c) $\cos(2x^2 - 1)$
(d) $\sin^3 2x$ (e) $\tan\left(\frac{x}{3} - 2\right)$ (f) $\sin \frac{x}{2} \cos 2x$
(g) $\frac{1 - \sin x}{1 + \sin x}$ (h) $x^2 \tan \frac{x}{2}$ (i) $\cos x^2$
(j) $x(\cos 2x - \sin x)$

3 If $y = \sin 2x$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and show that $\frac{d^2y}{dx^2} + 4y = 0$.

4 If $y = x \sin 2x$, find the value of $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$.

5 Given $y = A \cos 2x + B \sin 2x$, where A and B are constants, show that $\frac{d^2y}{dx^2} + 4y = 0$. If also $y = 3$ when $x = \frac{\pi}{2}$ and $\frac{dy}{dx} = 4$ when $x = 0$, find the value of A and of B.

6 If $y = \cos\theta + 2\sin\theta$, find the values of θ $(0 < \theta < 2\pi)$ for which $\frac{dy}{d\theta} = 0$.

7 Find $\frac{dy}{dx}$ if $y = (\sin x + \cos 2x)^2$.

8 Solve the equation $\frac{d}{dx}(x + \sin 2x) = 2$ for $0 < x < \pi$.

9 Differentiate $\frac{\sin x}{1 + \cos x}$ wrt x and hence find $\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \cos x}$.

10 (a) Show that if $y = 2 \sin x - \cos x$, then $\frac{dy}{dx} = 0$ when $\tan x = -2$.
Hence find the values of x $(0 < x < 2\pi)$ where y has stationary values.

(b) Find the value of x $(0 < x < 2\pi)$ for which $y = \frac{3 \cos x}{2 - \sin x}$ is stationary. Hence find the maximum and minimum values of y.

11 Find the equations of the tangents to the curve $y = \sin x$ where $x = 0$ and $x = \pi$.

12 Find the equation of the tangent to the curve $y = \cos x$ where $x = \frac{\pi}{2}$.

Integration of Trigonometric Functions

If $y = \sin(ax + b)$, then $\frac{dy}{dx} = a\cos(ax + b)$.

Therefore $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

If $y = \cos(ax + b)$, then $\frac{dy}{dx} = -a\sin(ax + b)$.

Therefore $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

If $y = \tan(ax + b)$, then $\frac{dy}{dx} = a\sec^2(ax + b)$.

Therefore $$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$

For all these results, x must be in radians.

Example 24

Integrate (a) $\int \sin 3x\,dx$, *(b)* $\int \cos \frac{x}{2}\,dx$.

(a) $\int \sin 3x\,dx = \frac{-\cos 3x}{3} + c = -\frac{1}{3}\cos 3x + c$

(b) $\int \cos \frac{x}{2}\,dx = \frac{\sin \frac{x}{2}}{\frac{1}{2}} + c = 2\sin \frac{x}{2} + c$

Example 25

Find the area of the shaded region in Fig.17.4 between the part OA of the curve $y = \sin x$ *and the line OA, where O is the origin and A is the point* $(\frac{\pi}{2}, 1)$.

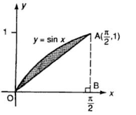

Fig. 17.4

The shaded area is the area under the curve minus the area of $\triangle$OBA where AB is perpendicular to x-axis.

$$\text{Area} = \int_0^{\frac{\pi}{2}} \sin x \, dx - \tfrac{1}{2} \times \tfrac{\pi}{2} \times 1$$

$$= \Big[-\cos x\Big]_0^{\frac{\pi}{2}} - \tfrac{\pi}{4}$$

$$= \left(-\cos \tfrac{\pi}{2}\right) - (-\cos 0) - \tfrac{\pi}{4} = 0 - (-1) - \tfrac{\pi}{4} = 1 - \tfrac{\pi}{4} \text{ units}^2$$

Example 26

Find $\int \sin^2 x \, dx$.

We cannot find $\int \sin^2 x \, dx$ directly as it is not in the form $\sin(ax + b)$.

We use the formula $\cos 2x = 1 - 2 \sin^2 x$ to convert it to a suitable form.

Then $\int \sin^2 x \, dx = \int \frac{(1 - \cos 2x)}{2} \, dx$

$$= \int \left(\tfrac{1}{2} - \tfrac{\cos 2x}{2}\right) dx = \tfrac{x}{2} - \tfrac{\sin 2x}{2 \times 2} + c = \tfrac{x}{2} - \tfrac{\sin 2x}{4} + c.$$

The same method is used to find $\int \cos^2 x \, dx$.

Example 27

Sketch the curve $y = 1 + \cos x$ for $0 \le x \le \pi$. This curve is rotated about the x-axis through 2π radians. Find the volume created in terms of π.

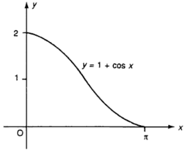

Fig. 17.5

Fig. 17.5 shows the curve, which is $y = \cos x$ moved up 1 unit.

The volume $= \int_0^{\pi} \pi(1 + \cos x)^2 \, dx$

$= \pi \int_0^{\pi} (1 + 2\cos x + \cos^2 x) \, dx$

$= \pi \int_0^{\pi} \left(1 + 2\cos x + \frac{1 + \cos 2x}{2}\right) dx$

$= \pi \int_0^{\pi} \left(\frac{3}{2} + 2\cos x + \frac{1}{2}\cos 2x\right) dx$

$= \pi \left[\frac{3x}{2} + 2\sin x + \frac{1}{4}\sin 2x\right]_0^{\pi}$

$= \pi\left(\frac{3\pi}{2} + 2\sin\pi + \frac{1}{4}\sin 2\pi\right) - \pi(0) = \frac{3\pi^2}{2}$ units^3

Exercise 17.7 *(Answers on page 642.)*

1 Integrate wrt x:

(a) $\sin 2x$ (b) $\cos 4x$ (c) $\sin \frac{x}{2}$

(d) $3\sin 3x$ (e) $\sec^2 3x$ (f) $\cos 2x - \sin x$

(g) $\sin x + \cos x$ (h) $\cos^2 \frac{x}{2}$ (i) $\cos 5x$

(j) $\sin\left(\frac{\pi}{4} - x\right)$ (k) $\sec^2 \frac{x}{2}$ (l) $\cos 2x - \sin x$

(m) $(\cos x - \sin x)^2$ (n) $2\sin x + \frac{1}{2}\sin 2x$

2 Evaluate

(a) $\int_0^{\frac{\pi}{2}} \cos x \, dx$ (b) $\int_0^{\frac{\pi}{2}} \sin x \, dx$

(c) $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ (d) $\int_0^{\frac{\pi}{4}} \sec^2 x \, dx$

(e) $\int_0^{\frac{\pi}{2}} (\sin 2x - \cos x) \, dx$ (f) $\int_0^{\frac{\pi}{2}} (\cos x + \sin x)^2 \, dx$

(g) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin \frac{3x}{2} \, dx$ (h) $\int_0^{\frac{\pi}{3}} \sin 3x \, dx$

(i) $\int_0^{\pi} \cos \frac{x}{2} \, dx$ (j) $\int_0^{\pi} \cos 2x \, dx$

(k) $\int_0^{\frac{5\pi}{12}} \cos^2 x \, dx$

3 If $\frac{dy}{d\theta} = \frac{1}{\theta^2} + \frac{1}{2}\cos 2\theta$, find y if $y = 1$ when $\theta = \frac{\pi}{2}$.

4 Find (a) the area of the region enclosed by the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ and (b) the volume created if this region is rotated about the x-axis.

5 Differentiate $\frac{1}{1 + \cos x}$ wrt x. Hence find the area of the region under the curve $y = \frac{\sin x}{(1 + \cos x)^2}$ between $x = 0$ and $x = \frac{\pi}{2}$.

6 The region bounded by the x-axis and the part of the curve $y = 2\sin x$ between $x = 0$ and $x = \pi$ is rotated about the x-axis through 360°. Find the volume of the solid generated.

7 Sketch the curves $y = \cos x$ and $y = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$.

Find (a) the value of x where the curves intersect, (b) the area of the region bounded by the two curves and the x-axis. (c) If this region is rotated about the x-axis through 360°, find the volume of the solid created.

8 Find the area of the region enclosed by the x-axis, the y-axis, the curve $y = \cos x$ and the line $x = \frac{\pi}{6}$. If this region is rotated about the x-axis through 360°, find the volume created.

9 Using an identity for $\cos 4x$, find $\int \cos^2 2x \, dx$.

10 Sketch the curves $y = \cos x$ and $y = \sin 2x$ for $0 \leq x \leq \frac{\pi}{2}$. Find

(a) the value of x where the curves intersect $\left(\text{apart from } x = \frac{\pi}{2}\right)$ and

(b) the area of the region enclosed by the two curves and the x-axis.

11 (a) Show that $\frac{1 - \cos 2x}{1 + \cos 2x} \equiv \sec^2 x - 1$.

(b) Hence find the value of $\int_0^{\frac{\pi}{4}} \frac{1 - \cos 2x}{1 + \cos 2x} \, dx$.

12 By writing $3x$ as $2x + x$, show that $\cos 3x \equiv 4\cos^3 x - 3\cos x$.

Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$.

SUMMARY

- $\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{(n + 1)a} + c \; (n \neq -1)$
- *Product rule:* If $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$
- *Quotient rule:* If $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
- For x in radians, a and b constants:

  $\frac{d}{dx}\sin(ax + b) = a\cos(ax + b)$, $\int \cos(ax + b)\, dx = \frac{1}{a}\sin(ax + b) + c$

  $\frac{d}{dx}\cos(ax + b) = -a\sin(ax + b)$, $\int \sin(ax + b)\, dx = -\frac{1}{a}\cos(ax + b) + c$

  $\frac{d}{dx}\tan(ax + b) = a\sec^2(ax + b)$, $\int \sec^2(ax + b)\, dx = \frac{1}{a}\tan(ax + b) + c$
- To integrate $\sin^2 x$ or $\cos^2 x$, use the identity for $\cos 2x$.

14 If $r = a\sqrt{1 - \cos\theta}$, show that $\frac{dr}{d\theta} = \frac{a}{\sqrt{2}}\cos\frac{\theta}{2}$.

15 (a) Given that $y = \sin^2 x \cos 2x$, find the values of x $(0 \le x \le \pi)$ for which $\frac{dy}{dx} = 0$.

(b) Find the values of x $(0 \le x \le 2\pi)$ for which $y = \frac{1 + \sin x}{\sin x + \cos x}$ is stationary. State the maximum and minimum values of y.

16 If $y = \sin 2\theta$, find the approximate change in y when θ is increased from $\frac{\pi}{6}$ to $\frac{\pi}{6} + 0.01$.

17 Show that the function $\frac{x}{x^2 - 1}$ is always decreasing for $x > 1$.

18 Find $\frac{dy}{dx}$ in terms of x and y if $2xy^2 + y + 2x = 8$. Hence find the gradient of the curve at the points where $x = 1$.

19 If $y = a \sin 2x$, where a is a constant, satisfies the equation $\frac{d^2y}{dx^2} + 8y = 4\sin 2x$, find the value of a.

20 Given that $r^2(1 + \cos\theta) = k$, where k is a constant, show that $\frac{dr}{d\theta} = \frac{r}{2}\tan\frac{\theta}{2}$.

B

21 Solve the equation $\int_x^{2x} \sin\frac{t}{2}\, dt = 0$ for $0 \le x \le 2\pi$.

22 Find $\frac{dy}{dx}$ for each of the functions $xy = a$ and $y = \sqrt{k^2 + x^2}$ where a and k are constants. Hence show that the tangents at the point of intersection of the curves are perpendicular.

23 A particle moves in a straight line and its distance s from a fixed point O of the line at time t is given by $s = 4\sin 2t$.

(a) Show that its velocity v and its acceleration a at time t are given by $v = 2\sqrt{16 - s^2}$ and $a = -4s$.

(b) Find the greatest distance from O reached by the particle.

24 At a certain port the height h metres of the tide above the low water level is given by $h = 2(1 + \cos\theta)$ where $\theta = \frac{\pi t}{450}$ and t is the time in minutes after high tide.

(a) What length of time is there between high and low tide?

(b) At what rate is the tide falling, in metres per minute, 75 minutes after high tide?

(c) A bridge is 10 metres above the low water level. A boat can only sail under this bridge when the distance between the water and the bridge is not less than 7 metres. How long after high tide will it be before the boat can sail under the bridge?

25 (a) Differentiate $\cot\theta$ wrt θ.

(b) A cone has a base radius r and a semi-vertical angle θ. Show that its volume $V = \frac{1}{3}\pi r^3 \cot\theta$.

(c) r is fixed but θ is measured as 45° with an error of 4%. Find the percentage error in the calculated value of V.