# Conversion to Linear Form

# 16

In science, when two variables $x$ and $y$ are thought to be connected, a set of measurements is made. The results can be used to find the mathematical law connecting $x$ and $y$ — if there is one. When the law is found, it can be used to predict further values and these can be tested by other experiments to see if the law is still valid.

Usually the results are plotted as a graph. If this is a straight line, the relationship is easily deduced as it will be of the form $y = mx + c$, and $m$ and $c$ can be found from the graph.

However if the graph is not a straight line, the relationship will not be so simple. A trial formula is therefore guessed. We convert this formula to a linear form and see if the transformed values lie on a straight line graph. If they do, then we can confirm that the formula is true for these values, allowing for experimental errors.

Two very common relationships are $\boldsymbol{y = ab^x}$ and $\boldsymbol{y = ax^b}$ where $a$ and $b$ are constants.

### Example 1

*The following set of measurements of two variables x and y were obtained in an experiment. It is thought that they are related by the formula $y = ab^{-x}$. By converting this to a linear form, find whether the relationship is true for these values.*

| $x$ | 1.5 | 2.8 | 3.0 | 4.2 | 5.0 | 6.5 |
|---|---|---|---|---|---|---|
| $y$ | 80 | 35 | 33 | 18 | 10 | 6 |

If the formula is $y = ab^{-x}$, then taking the lg of each side, $\lg y = \lg a - x \lg b$.

Now write $Y = \lg y$.

Then $Y = -x \lg b + \lg a$, which is a linear equation of the form $Y = mx + c$, where $m = -\lg b$ and $c = \lg a$.

So we plot values of $Y$ (= $\lg y$) against $x$.

First we find the values of $Y$.

| $x$ | 1.5 | 2.8 | 3.0 | 4.2 | 5.0 | 6.5 |
|---|---|---|---|---|---|---|
| $Y$ (= $\lg y$) | 1.90 | 1.54 | 1.52 | 1.26 | 1 | 0.78 |

These values are plotted as shown in Fig.16.1. We see that the points lie very nearly in a straight line. Any inaccuracies can reasonably be assumed to be due to experimental errors. We draw the line which fits the points as well as we can judge. There may be some difference of opinion over the position of the line, so our results will be approximate.

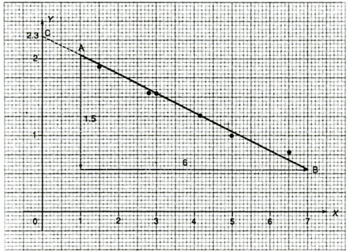


*Fig. 16.1*

Now find the gradient by taking two well spaced points such as A and B on the line. It helps to make the $x$-step between these points a convenient number. The gradient $= \frac{-1.5}{6} = -0.25$.

Then $-\lg b = -0.25$ and $b \approx 1.8$.

To find $c = \lg a$, extend the line to cut the $Y$-axis (point C).
Then $\lg a \approx 2.3$ giving $a \approx 200$.

Hence we find that the law relating these values is $y = 200 \times 1.8^{-x}$.

## Example 2

*The following set of values for two variables x and y was obtained in an experiment. It is believed that they are related by the formula $y = ax^b$. By converting to a linear form, estimate the values of a and b. From your graph, estimate the value of x for which y = 2000 and compare with the value found using the formula.*

| $x$ | 20 | 30 | 40 | 50 |
|---|---|---|---|---|
| $y$ | 890 | 1640 | 2500 | 3700 |

If the formula is $y = ax^b$, then taking the lg of each side, $\lg y = \lg a + b \lg x$.

We write $Y = \lg y$ and $X = \lg x$.

Then $Y = bX + \lg a$ which is a linear equation of the form $y = mx + c$ where $m = b$ and $c = \lg a$.

If we plot values of $Y$ (= lg $y$) and $X$ (= lg $x$) and the graph is a straight line, then the relationship is correct.

Now find the values of $X$ (= lg $x$) and $Y$ (= lg $y$).

| $X$ | 1.30 | 1.48 | 1.60 | 1.70 |
|---|---|---|---|---|
| $Y$ | 2.95 | 3.21 | 3.40 | 3.57 |

These are plotted as shown in Fig.16.2. To allow space for large scales, we take $X$ from 1.3 and $Y$ from 2.9.

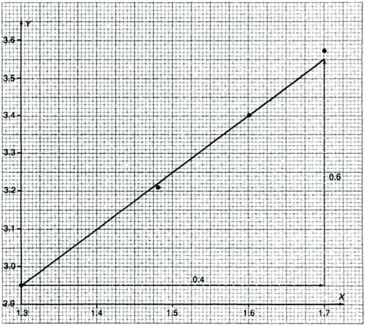


*Fig. 16.2*

### Example 5

*Fig.16.7 shows the straight line obtained by plotting lg y against lg x. Find*

*(a) lg y in terms of lg x,*
*(b) y in terms of x,*
*(c) the value of x when y = 700.*

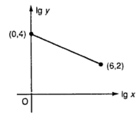


*Fig. 16.7*

(a) The gradient of the line = $\frac{-2}{6} = -\frac{1}{3}$ and the intercept on the lg $y$ axis is 4. Hence the equation of the line is lg $y = -\frac{1}{3}$ lg $x + 4$ which is the expression required.

(b) From (a), lg $y = -\frac{1}{3}$ lg $x + 4$ i.e. 3 lg $y$ + lg $x$ = 12.

Then lg $y^3x$ = lg $10^{12}$ giving $y^3x = 10^{12}$ or $y^3 = \frac{10^{12}}{x}$.

Hence $y = \frac{10^4}{\sqrt[3]{x}} = 10^4x^{-\frac{1}{3}}$.

(c) If $y = 700$, then $700 = 10\,000x^{-\frac{1}{3}}$ and $x^{\frac{1}{3}} = \frac{100}{7}$.

Hence $x \approx 2915$.

### Example 6

*Convert each of the following relations to a linear form and state what functions of x and y should be plotted to obtain a straight line graph. State also the gradient and intercept of the straight line in terms of a and b.*

*(a)* $\frac{a}{x} + \frac{b}{y} = 2$ *(b)* $y = ax + \frac{b}{x}$ *(c)* $y^2 = a + bx$

*(d)* $y = \frac{a}{x-b}$ *(e)* $y = a(1.5)^{-bx}$ *(f)* $y = a(x+3)^b$

(a) Take $X = \frac{1}{x}$, $Y = \frac{1}{y}$. Then $aX + bY = 2$ i.e. $Y = -\frac{a}{b}X + \frac{2}{b}$.

Plot $Y$ against $X$. Gradient = $-\frac{a}{b}$, intercept = $\frac{2}{b}$.

(b) If $y = ax + \frac{b}{x}$, then $xy = ax^2 + b$. Take $X = x^2$, $Y = xy$.

This gives the linear equation $Y = aX + b$. Plot $Y$ against $X$.
Gradient = $a$, intercept = $b$.

(c) Take $Y = y^2$. Then $Y = bx + a$. Plot $Y$ against $x$. Gradient = $b$, intercept = $a$.

(d) $xy - by = a$ so $by = xy - a$ or $y = \frac{1}{b}xy - \frac{a}{b}$. Plot $y$ against $X = xy$. Gradient = $\frac{1}{b}$, intercept = $-\frac{a}{b}$.

(e) $y = a(1.5)^{-bx}$. Then $\lg y = \lg a - bx \log 1.5$. Take $Y = \lg y$ then $Y = -(b \lg 1.5)x + \lg a$. Plot $Y$ against $x$. Gradient = $-b \lg 1.5$ and intercept = $\lg a$.

(f) $y = a(x + 3)^b$. Then $\lg y = \lg a + b \lg(x + 3)$. Take $Y = \lg y$ and $X = \lg(x + 3)$. Plot $Y$ against $X$. Gradient = $b$ and intercept = $\lg a$.

## Exercise 16.1 *(Answers on page 640.)*

**1** A set of values of $x$ and $y$ are believed to be connected by the equation $y = ab^x$ where $a$ and $b$ are constants. Values of $x$ and $\lg y$ are plotted and the graph is a straight line with gradient 0.47 and intercept –0.65. Find the value of $a$ and of $b$ correct to 2 significant figures.

**2** A graph of $\lg y$ against $\lg x$ gives a straight line with gradient 3 and intercept 1.3. Find $y$ in terms of $x$.

**3** Fig. 16.8 shows the graph of $\lg y$ against $\lg x$, where $y = ax^b$. Find the value of $a$ and of $b$.

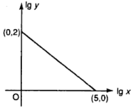


*Fig. 16.8*

**4** Covert the equation $by = ax^2 + x$ into the linear form $Y = mX + c$, stating $X$ and $Y$ in terms of $x$ and $y$. $Y$ is plotted against $X$ and the graph has a gradient of 2.3 with intercept 0.5. Find the value of $a$ and of $b$.

**5** The following results were obtained experimentally for two variables $x$ and $y$:

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 42 | 120 | 430 | 920 | 2600 |

It is believed that $x$ and $y$ are related by the equation $y = ab^x$. By drawing a straight line graph, verify this is confirmed by the given data, except for one point. Using your graph estimate the value of $a$ and of $b$ and calculate a more accurate value of $y$ for the point which did not fit.

6 The variables $x$ and $y$ are related in such a way that when $\frac{1}{x+1}$ is plotted against $y$, a straight line is obtained, passing through the points (1, 5) and (3, 11) (Fig. 16.9). Find $y$ in terms of $x$.

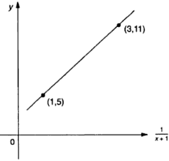


*Fig. 16.9*

7 It is believed that two variables $u$ and $v$ are related by the equation $uv^2 = av + b$, where $a$ and $b$ are constants.
A set of values of $u$ and $v$ was obtained, as in the following table:

| | | | | |
|---|---|---|---|---|
| $v$ | 1 | 2 | 5 | 8 |
| $u$ | 12 | 3.5 | 0.8 | 0.41 |

By plotting $uv^2$ against $v$, verify that these values satisfy the equation and find approximate values for $a$ and $b$.

8 Two variables $x$ & $y$ are connected by the equation $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$. Given the following values of x and y, show how a straight line graph may be drawn

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 |
| $y$ | 6.5 | 6.01 | 6.06 | 6.25 | 6.48 |

Draw this graph and from it, estimate the value of $a$ and of $b$.

9 The following set of values of $x$ and $y$ obtained in an experiment are thought to be connected by the equation $\frac{p}{y} - \frac{q}{x^2} = 1$.

| | | | | |
|---|---|---|---|---|
| $x$ | 1.5 | 2 | 3.4 | 5 |
| $y$ | 0.55 | 1 | 2.5 | 4.7 |

Explain how a straight line graph may be obtained and draw this graph for these values. From your graph, estimate the value of $p$ and of $q$.

**10** The following table gives a set of related values of $x$ and $y$:

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 1.2 | 1.5 | 2 | 2.5 | 3 |
| $y$ | 19 | 11 | 5 | 3 | 1 |

$x$ and $y$ are known to be related by the equation $x^2y = p + qx^2$. Convert this equation to linear form and draw a graph for the given values of $x$ and $y$. Using the graph, find approximate values for $p$ and $q$.

## SUMMARY

- If $y = ax^b$, $\lg y = b \lg x + \lg a$

  Linear form $Y = bX + \lg a$

  where $Y = \lg y$, $X = \lg x$.

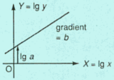


Fig. 16.10

- If $y = ab^x$, $\lg y = x(\lg b) + \lg a$

  Linear form $Y = (\lg b)x + \lg a$

  where $Y = \lg y$.

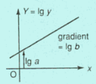


Fig. 16.11

- Other relationships can also be converted to linear form, e.g.

  $\frac{a}{y} + \frac{b}{x} = 1$ gives $Y = -\frac{b}{a}X + \frac{1}{a}$ where $Y = \frac{1}{y}$, $X = \frac{1}{x}$.

  $ay^2 = bx^2 + x$ gives $\frac{y^2}{x} = \frac{b}{a}x + \frac{1}{a}$ i.e. $Y = \frac{b}{a}x + \frac{1}{a}$ where $Y = \frac{y^2}{x}$.

## REVISION EXERCISE 16 *(Answers on page 640.)*

**1** Corresponding values of $x$ and $y$ are showing in the following table:

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 2 | 3 | 5 | 6 | 9 |
| $y$ | 1.7 | 2.2 | 3.0 | 3.3 | 4.1 |

It is known that $x$ and $y$ are related by the equation $y^2 = a + bx$. Show that a linear equation can be derived from this and draw its graph for the above values. Hence estimate the value of $a$ and of $b$ and estimate the smallest possible value of $x$.

**2** (a) Fig. 16.12 shows part of the straight line obtained by plotting $y$ against $\frac{1}{x^2}$. Two of the points on the line are given. Find $y$ in terms of $x$.

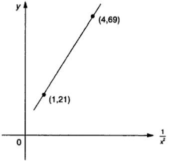


*Fig. 16.12*

(b) lg $y$ is plotted against lg $x$ and a straight line obtained, part of which is shown in Fig. 16.13. Two of the points on the line are given. Express $y$ in terms of $x$.

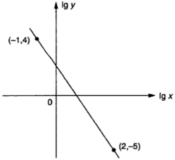


*Fig. 16.13*

(c) If variables $x$ and $y$ are connected by the equation $ax^2 - y^2 = bx$ ($a$ and $b$ constants) explain how the value of $a$ and of $b$ can be obtained from a straight line graph.

**3** Measured values of $x$ and $y$ are given in the following table.

| $x$ | 1 | 2 | 2.5 | 5 | 8 |
|---|---|---|---|---|---|
| $y$ | 0.20 | 1.16 | 2.03 | 9.02 | 23.96 |

It is known that $x$ and $y$ are related by the equation $ax^2 + by = x$.
Explain how a straight line graph may be drawn to represent the given equation and draw it for the given data.
Use your graph to estimate the value of $a$ and of $b$. (C)

**4** (a) It is known that the variables $x$ and $y$ are related by the equation $y = \frac{p}{x - q}$, where $p$ and $q$ are unknown constants.

Express this equation in a form suitable for drawing a straight line graph, and state which variable should be used for each axis. Explain how the value of $p$ and of $q$ could be determined from this graph.

(b) The table shows experimental values of two variables $x$ and $y$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 |
| $y$ | 14.6 | 6.8 | 4.0 | 2.4 | 1.2 | 0.4 |

It is known that $x$ and $y$ are related by an equation of the form $y = ax + \frac{b}{x}$ where $a$ and $b$ are unknown constants. Plot $xy$ against $x^2$ and use the graph to estimate (i) the value of $a$ and of $b$, (ii) the value of $y$ when $x = 1.2$. (C)

**5** (a) The table shows experimental values of two variables $x$ and $y$:

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
| $y$ | 1.8 | 2.1 | 2.4 | 2.6 | 2.9 | 3.1 |

It is known that $x$ and $y$ are related by the equation $y = kx^n$, where $k$ and $n$ are constants. Draw a suitable straight line graph to represent the above data and use it to estimate $k$ and $n$.

(b) The variables $x$ and $y$ are related in such a way that when $x + y$ is plotted against $x^2$ a straight line is obtained passing through (1, −1) and (5, 2) (Fig. 16.14). Find (i) the values of $x$ when $x + y = 5$, (ii) $y$ as a function of $x$, (iii) the values of $x$ when $y = 0$. (C)

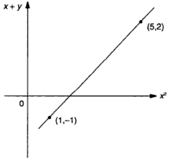


*Fig. 16.14*

**6** (a) The variables $x$ and $y$ are connected by the equation $y = ax^b$ where $a$ and $b$ are constants. Fig. 16.15 shows the straight line graph obtained by plotting lg $y$ against lg $x$.
Calculate the value of $a$ and of $b$ and hence find the value of $y$ when $x = 5$.

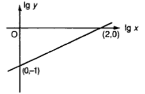

*Fig. 16.15*

(b) The variables $x$ and $y$ are connected by the equation $x + py = qxy$, where $p$ and $q$ are unknown constants. Explain how the value of $p$ and of $q$ may be obtained from a suitable straight line graph. (C)

**7** It is predicted from theory that two variables $P$ and $T$ are related by the equation $P = a + \frac{b}{\sqrt{(T-3)}}$.
The following values of $P$ and $T$ were found by experiment:

| $T$ | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|
| $P$ | 13.3 | 10.6 | 9.4 | 8.9 | 8.7 |

By plotting $P$ against $\frac{1}{\sqrt{(T-3)}}$, confirm that the equation is approximately true for these values. Use your graph to estimate the value of $a$ and of $b$.

**8** (a) Variables $x$ and $y$ are known to be related by an equation of the form $a(x + y - b) = bx^2$, where $a$ and $b$ are constants. Observed values of the two variables are shown in the following table.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 0.5 | 0.5 | 1.5 | 3.5 | 6 |

Plot $x + y$ against $x^2$, draw the straight line graph and use it to estimate the value of $a$ and of $b$.

(b) Variables $x$ and $y$ are related by the equation $\frac{x^2}{p^2} + \frac{2y^2}{q^2} = 1$ where $p$ and $q$ are positive constants.

When the graph of $y^2$ against $x^2$ is drawn, a straight line is obtained. Given that the intercept on the $y^2$-axis is 4.5 and that the gradient of the line is –0.18, calculate the value of $p$ and of $q$. (C)