Exponential and Logarithmic Functions

15

In this chapter, we study functions such as 3^x where x is an index or **exponent.** Hence 3^x is called an **exponential function** and 3 is the **base** of the function.

First, here is a revision check on the rules for working with indices.

RULES FOR INDICES

The three basic rules are:

I $x^m \times x^n = x^{m+n}$	When multiplying exponential functions with the same base, ADD the indices.
II $x^m \div x^n = x^{m-n}$	When dividing exponential functions with the same base, SUBTRACT the indices.
III $(x^m)^n = x^{mn}$	If an exponential function is raised to another power, MULTIPLY the indices.

Using these rules we can deduce that $x^0 = 1$ $(x \neq 0)$ and that $x^{-m} = \frac{1}{x^m}$ $(x \neq 0)$. A negative index gives the reciprocal of the function.

Fractional indices

We can also find a meaning for fractional indices. For example

$(9^{\frac{1}{2}})^2 = 9^{\frac{1}{2} \times 2} = 9^1 = 9$. So $9^{\frac{1}{2}} = \sqrt{9} = 3$.

In general, $x^{\frac{1}{n}} = \sqrt[n]{x}$.

Further, $27^{\frac{2}{3}} = (27^{\frac{1}{3}})^2 = (\sqrt[3]{27})^2 = (3)^2 = 9$.

In general, $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$.

Example 1

Find the values of (a) $100^{\frac{3}{2}}$ *(b)* $32^{-\frac{2}{5}}$.

(a) $100^{\frac{3}{2}} = (\sqrt{100})^3 = (10)^3 = 1000$

(b) $32^{-\frac{2}{5}} = \frac{1}{32^{\frac{2}{5}}} = \frac{1}{(\sqrt[5]{32})^2} = \frac{1}{(2)^2} = \frac{1}{4}$.

Example 2

Show that (a) $3^{2x-1} = \frac{9^x}{3}$, *(b)* $2^{x-1} \times 8^{x+1} = 4^{2x+1}$

(a) $3^{2x-1} = 3^{2x} \times 3^{-1} = (3^2)^x \times \frac{1}{3} = \frac{9^x}{3}$

(b) $2^{x-1} \times 8^{x+1} = 2^{x-1} \times (2^3)^{x+1}$

$= 2^{x-1} \times 2^{3x+3}$

$= 2^{4x+2} = (2^2)^{2x+1} = 4^{2x+1}$

Exercise 15.1 *(Answers on page 638.)*

1 Find the value of:

(a) 8^0 (b) 4^{-2} (c) $(-3)^{-3}$

(d) $8^{\frac{1}{3}}$ (e) $81^{\frac{3}{4}}$ (f) $32^{\frac{2}{5}}$

(g) $16^{-\frac{3}{4}}$ (h) $(x^3)^{-\frac{2}{3}}$ (i) $\left(-\frac{8}{27}\right)^{-\frac{1}{3}}$

2 Show that

(a) $2^{x+2} = 4 \times 2^x$ (b) $16^x = 4^{2x}$ (c) $2^{2x+3} = 8 \times 4^x$

(d) $2^{3-x} = \frac{8}{2^x}$ (e) $4^{2-x} = \frac{16}{2^{2x}}$ (f) $5^x \times 25^y = 5^{x+2y}$

3 Simplify (a) $2^{2-2x} \times 4^{x-1}$, (b) $27^{\frac{x}{3}} \div 9^{\frac{x}{2}}$, (c) $8^{\frac{2x}{3}} \times 4^{2-x}$.

EXPONENTIAL EQUATIONS

An equation such as $3^x = 81$ is an **exponential equation.** The unknown (x) is the exponent. We can solve such equations by expressing both sides in terms of the *same base.* Sometimes this can be done directly. If not, a more general method using logarithms can be used. This will be shown later.

Example 6

Solve the equation $2^x + 2^{1-x} = 3$.

The equation is $2^x + 2^1 \times 2^{-x} = 3$ i.e. $2^x + \frac{2}{2^x} = 3$.

Take $p = 2^x$.

The equation becomes $p + \frac{2}{p} = 3$ i.e. $p^2 - 3p + 2 = 0$.

Then $(p - 2)(p - 1) = 0$ and $p = 2$ or 1.

Then $2^x = 2 = 2^1$ and $x = 1$ or $2^x = 1 = 2^0$ and $x = 0$.

The solutions are $x = 1$ or 0.

Example 7

Solve the simultaneous equations

$$3^x \times 9^y = 1 \qquad \text{(i)}$$

and

$$2^{2x} \times 4^y = \frac{1}{8} \qquad \text{(ii)}$$

In equation (i), we see that each term can be expressed as a power of 3.

Then $3^x \times (3^2)^y = 3^0$

so $x + 2y = 0$ (iii)

Similarly, each term of equation (ii) can be expressed as a power of 2.

Then $2^{2x} \times (2^2)^y = 2^{-3}$

so $2x + 2y = -3$ (iv)

Solving equations (iii) and (iv), we obtain $x = -3$, $y = 1\frac{1}{2}$.

Exercise 15.2 *(Answers on page 638.)*

1 Solve the following equations:

(a) $2^x = 64$
(b) $5^x = 125$
(c) $5^x = 1$
(d) $9^x = 81$
(e) $16^x = 0.125$
(f) $4^x = 0.5$
(g) $9^x = \frac{1}{729}$
(h) $2^{2x} - 9(2^x) + 8 = 0$
(i) $3^{2x} - 12(3^x) + 27 = 0$
(j) $5^{2x} + 1 = 26(5^{x-1})$
(k) $2^{2x+1} - 129(2^x) + 64 = 0$
(l) $3^{2x+1} + 26(3^x) - 9 = 0$
(m) $2^{2x+1} - 15(2^x) = 8$
(n) $2^x + 2^{2-x} = 5$
(o) $2^{x+3} = 2^{1-x} + 15$
(p) $3^{2x+1} - 28(3^{x-1}) + 1 = 0$

2 Show that the equation $2^{2x} + 3(2^{x+1}) + 8 = 0$ has no solutions.

Conversely, if $\log_x 10 = 4$, then $10 = x^4$
if $\log_3 x = 5$, then $x = 3^5$.

Example 8

Write in logarithmic form: (a) $3^2 = 9$, (b) $x^3 = 10$, (c) $2^{-2} = \frac{1}{4}$.

(a) If $3^2 = 9$, then $\log_3 9 = 2$.
(b) If $x^3 = 10$, then $\log_x 10 = 3$.
(c) If $2^{-2} = \frac{1}{4}$, then $\log_2\left(\frac{1}{4}\right) = -2$. (Logarithms can be negative)

Example 9

Write in exponential form: (a) $4 = \log_3 x$, (b) $x = \log_5 7$, (c) $2 = \log_x 5$.

(a) $4 = \log_3 x$ becomes $3^4 = x$
(b) $x = \log_5 7$ becomes $5^x = 7$
(c) $2 = \log_x 5$ becomes $x^2 = 5$

Example 10

Find the value of x if (a) $x = \log_2 64$, (b) $\log_x 25 = 2$, (c) $x = \log_3\left(\frac{1}{3}\right)$, (d) $\log_3 x = 4$.

(a) If $x = \log_2 64$, then $2^x = 64$ and $x = 6$ (as $64 = 2^6$).
(b) If $\log_x 25 = 2$, then $x^2 = 25$ and $x = 5$ (+ as base must be positive).
(c) If $x = \log_3\left(\frac{1}{3}\right)$, then $3^x = \frac{1}{3} = 3^{-1}$ and $x = -1$.
(d) If $\log_3 x = 4$, then $3^4 = x = 81$.

Exercise 15.4 *(Answers on page 638.)*

1 Write in logarithmic form:

(a) $4^2 = 16$	(b) $3^3 = 81$	(c) $10^3 = 1000$
(d) $10^{-3} = 0.001$	(e) $4^{\frac{1}{2}} = 2$	(f) $x^3 = 2$
(g) $7^x = 21$	(h) $x^{-4} = 16$	(i) $10^{-1} = 0.1$
(j) $8^2 = 64$	(k) $4^x = 9$	(l) $x^{-3} = 0.3$

2 Write the following in exponential form and hence find the value of x:

(a) $x = \log_2 16$	(b) $x = \log_3 27$
(c) $x = \log_4 64$	(d) $x = \log_2\left(\frac{1}{8}\right)$
(e) $x = \log_{10} 0.001$	(f) $x = \log_{64} 4$
(g) $x = \log_7\left(\frac{1}{49}\right)$	(h) $\log_5 625 = x$
(i) $x = \log_3\left(\frac{1}{27}\right)$	(j) $x = \log_{13} 169$
(k) $x = \log_{169} 13$	

3 Find the value of:

(a) $\log_5 5$ (b) $\log_8 64$ (c) $\log_3 1$

(d) $\log_a 1$ (e) $\log_4 16$ (f) $\log_3 243$

(g) $\log_9 3$ (h) $\log_p p^2$ (i) $\log_5 \frac{1}{25}$

(j) $\log_4 8$ (k) $\log_{16} 8$ (l) $\log_4 16$

(m) $\log_x x$ (n) $\log_x \left(\frac{1}{x}\right)$ (o) $\log_2 \left(\frac{1}{4}\right)$

4 Find the value of x if:

(a) $\log_x 9 = 2$ (b) $\log_2 x = -3$

(c) $\log_x 125 = 3$ (d) $\log_{x+1} 27 = 3$

(e) $\log_4 (x - 2) = 3$ (f) $\log_{2x} 36 = 2$

(g) $\log_x 81 = 4$ (h) $\log_3 x = -1$

(i) $\log_{x-2} 3 = 1$ (j) $\log_3 (x - 2) = 4$

(k) $\log_{2x} 64 = 3$

The Graph of the Logarithmic Function

As the logarithmic function is the inverse of the exponential function $y = a^x$, we can obtain its graph by reflecting $y = a^x$ in the line $y = x$ (Fig.15.3).

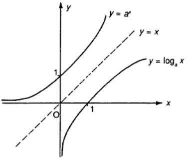

Fig.15.3

Note the following:

(1) $\mathbf{\log_a 1 = 0}$. This follows because $a^0 = 1$.
(2) $\log_a x$ is not defined if $x < 0$. The logarithm of a negative number does not exist.
(3) If $0 < x < 1$, $\log_a x < 0$. The logarithm of a positive number < 1 is always negative.
(4) $\log_a 0$ is undefined.
(5) As x increases, $\log_a x$ increases.

Rules for Logarithms

These are similar to the rules for indices.

Let $P = a^m$. Then $m = \log_a P$.
Let $Q = a^n$. Then $n = \log_a Q$.

I $PQ = a^m \times a^n = a^{m+n}$

Then $\log_a PQ = m + n = \log_a P + \log_a Q$.

$$\log_a PQ = \log_a P + \log_a Q$$

For example, $\log_a 12 = \log_a (4 \times 3) = \log_a 4 + \log_a 3$.

Note: Do NOT write $\log_a (P + Q) = \log_a P + \log_a Q$. This is not true.

II $\frac{P}{Q} = \frac{a^m}{a^n} = a^{m-n}$.

Then $\log_a \frac{P}{Q} = m - n = \log_a P - \log_a Q$.

$$\log_a \frac{P}{Q} = \log_a P - \log_a Q$$

For example, $\log_a 3 = \log_a 12 - \log_a 4$.

Note: This rule does NOT apply to $\frac{\log_a P}{\log_a Q}$ which is the division of two logarithms.

III $P^n = (a^m)^n = a^{mn}$

Then $\log_a P^n = mn = n \log_a P$

$$\log_a P^n = n \log_a P$$

For example, $\log_a 2^3 = 3 \log_a 2$ and $\log_a \sqrt{3} = \log_a 3^{\frac{1}{2}} = \frac{1}{2}\log_a 3$.

Two Special Logarithms

1 For any base, $a^0 = 1$. Hence

$$\log_a 1 = 0$$

The logarithm of 1 is always 0.

2 $a^1 = a$. Hence

$$\log_a a = 1$$

The logarithm of the base is always 1.

Example 11

Simplify (a) $\log_7 49$, (b) $\log_3 (\frac{1}{9})$.

(a) $\log_7 49 = \log_7 7^2 = 2 \log_7 7 = 2$ as $\log_7 7 = 1$

(b) $\log_3 \left(\frac{1}{9}\right) = \log_3 1 - \log_3 9 = 0 - \log_3 3^2$
$= -2 \log_3 3$
$= -2$

Example 12

Simplify $\log_4 9 + \log_4 21 - \log_4 7$.

$\log_4 9 + \log_4 21 - \log_4 7 = \log_4 (9 \times 21 \div 7) = \log_4 27 = \log_4 3^3 = 3 \log_4 3$

Example 13

Given that $\log_5 2 = 0.431$ and $\log_5 3 = 0.683$, find the value of
(a) $\log_5 6$, (b) $\log_5 1.5$, (c) $\log_5 8$, (d) $\log_5 12$, (e) $\log_5 \frac{1}{18}$.

We must express each in terms of powers of 2 and 3.

(a) $\log_5 6 = \log_5 (3 \times 2) = \log_5 3 + \log_5 2 = 1.114$

(b) $\log_5 1.5 = \log_5 \left(\frac{3}{2}\right) = \log_5 3 - \log_5 2 = 0.252$

(c) $\log_5 8 = \log_5 2^3 = 3 \log_5 2 = 1.293$

(d) $\log_5 12 = \log_5 (4 \times 3) = \log_5 4 + \log_5 3 = \log_5 2^2 + \log_5 3$
$= 2 \log_5 2 + \log_5 3 = 1.545$

(e) $\log_5 \frac{1}{18} = \log_5 1 - \log_5 18 = 0 - \log_5 (9 \times 2)$
$= -\log_5 9 - \log_5 2$
$= -\log_5 3^2 - \log_5 2$
$= -2 \log_5 3 - \log_5 2 = -1.797$

(With practice, some of these steps could be omitted).

Example 14

Given that $\log_a x = p$ and $\log_a y = q$, express
(a) $\log_a xy^2$, (b) $\log_a \frac{x}{y^3}$, (c) $\log_a \sqrt{\frac{ax^2}{y}}$ in terms of p and q.

(a) $\log_a xy^2 = \log_a x + \log_a y^2 = \log_a x + 2 \log_a y = p + 2q$

(b) $\log_a \frac{x}{y^3} = \log_a x - \log_a y^3 = \log_a x - 3 \log_a y = p - 3q$

Example 17

(a) If $\log_2 p = x$ and $\log_4 q = y$, express p^2q and $\frac{p}{q^2}$ as powers of 2.

(b) If also $p^2q = 16$ and $\frac{p}{q^2} = \frac{1}{32}$, find the values of x and y.

(a) First, we express p and q as powers of 2.

If $\log_2 p = x$, then $p = 2^x$.

If $\log_4 q = y$, then $q = 4^y = (2^2)^y = 2^{2y}$.

$p^2q = (2^x)^2 \times (2^{2y}) = 2^{2x+2y}$

$\frac{p}{q^2} = \frac{2^x}{(2^{2y})^2} = 2^{x-4y}$

(b) If $p^2q = 2^{2x+2y} = 16 = 2^4$,

then $2x + 2y = 4$

or $x + y = 2$ (i)

If $\frac{p}{q^2} = 2^{x-4y} = \frac{1}{32} = 2^{-5}$,

then $x - 4y = -5$ (ii)

Solving equations (i) and (ii), $x = \frac{3}{5}$ and $y = \frac{7}{5}$.

Example 18

Find the value of x if $\log_a x$, $\log_a (x+3)$ and $\log_a (x+12)$ are three consecutive terms of an AP.

As the terms are consecutive, $\log_a x + \log_a (x + 12) = 2 \log_a (x + 3)$.

Then $\log_a x(x + 12) = \log_a (x + 3)^2$ i.e. $x(x + 12) = (x + 3)^2$

which gives $x^2 + 12x = x^2 + 6x + 9$ or $6x = 9$.

Hence $x = 1\frac{1}{2}$.

Exercise 15.5 *(Answers on page 639.)*

1 Taking scales of 2 cm for 1 unit on each axis, draw the graph of $y = 2^x$ for $-2 \le x \le 2$. Add the line $y = x$. By reflecting $y = 2^x$ in this line, draw the graph of $y = \log_2 x$ for $\frac{1}{4} \le x \le 4$.

From your graph, find approximate values for (a) $\log_2 1.5$, (b) $\log_2 3$.

2 Simplify:

(a) $\log_7 \sqrt{2}$	(b) $\log_6 36$
(c) $\log_5 27$	(d) $\log_2 25$
(e) $\log_3 125$	(f) $\log_{10} 10\,000$
(g) $\log_6 121$	(h) $\log_x x^4$

(i) $\log_5 12.5 + \log_5 10$
(j) $2 \log_7 9 - \log_7 81$
(k) $\log_3 24 + \log_3 15 - \log_3 10$
(l) $\log_7 98 - \log_7 30 + \log_7 15$
(m) $\log_5 \sqrt{6} + \log_5 9$
(n) $\dfrac{\log_a x^3}{\log_a x^2}$
(o) $\log_4 8$
(p) $\log_9 81$
(q) $\log_3 5^3$
(r) $\log_4 4^x$
(s) $\log_3 \sqrt{10}$
(t) $\log_5 10$

3 Given that $\log_3 4 = 1.262$ and that $\log_3 5 = 1.465$, find the values of:
(a) $\log_3 20$
(b) $\log_3 0.8$
(c) $\log_3 1.25$
(d) $\log_3 100$
(e) $\log_3 64$
(f) $\log_3 80$
(g) $\log_3 6.25$
(h) $\log_3 15$
(i) $\log_3 0.25$

4 Given that $\log_7 2 = 0.356$ and $\log_7 3 = 0.565$, find the values of
(a) $\log_7 6$
(b) $\log_7 9$
(c) $\log_7 18$
(d) $\log_7 24$
(e) $\log_7 4.5$
(f) $\log_7 \frac{2}{3}$
(g) $\log_7 \sqrt{3}$
(h) $\log_7 14$
(i) $\log_7 42$
(j) $\log_7 \frac{3}{7}$
(k) $\log_7 4\frac{2}{3}$

5 If $\log_2 x^3 + \log_2 x = 8$, find the value of x.

6 If $\log_5 x = a$ and $\log_5 y = b$ express (a) xy^3, (b) $\frac{5x}{y}$ in terms of a and b.

7 If $\log_a x = p$ and $\log_a y = q$, express (a) $\log_a x^2y$, (b) $\log_a \sqrt{xy}$, (c) $\log_a \frac{x^3}{y}$, (d) $\log_a \frac{a^3x}{y^2}$ in terms of p and q.

8 Find y if $\log_3 y = 2 \log_3 7$.

9 If $\log_5 p - \log_5 4 = 2$, find the value of p.

10 Given that $\log_x 8 + \log_x 4 = 5$, find the value of x.

11 Given that $\log_3 x = p$ and $\log_9 y = q$, express (a) x^3y, (b) $\frac{x}{\sqrt{y}}$ as powers of 3.

12 Given that $\log_3 4 = p$ and $\log_3 5 = q$, find the value of x if (a) $\log_3 x = p + 2q$, (b) $\log_3 x = 2p - q + 2$.

13 $\log_2 x = a$ and $\log_4 y = b$. Express x^2y and $\frac{x^3}{y}$ as powers of 2.
Given also that $x^2y = 32$ and that $\frac{x^3}{y} = \frac{1}{8}$, find the values of a and b.

14 Given that $\log_{10} 2 = h$ and $\log_{10} 7 = k$, find the value of x if (a) $\log_{10} x = 2h + k$, (b) $\log_{10} x = 3h - k + 1$.

15 If $\log_{10} x = a$ and $\log_{10} y = b$, express $\log_{10} \sqrt{\frac{10x}{y^3}}$ in terms of a and b.

16 Given that $\log_a x^2y = p$ and that $\log_a\left(\frac{x}{y^2}\right) = q$, find $\log_a x$ and $\log_a y$ in terms of p and q and hence express $\log_a xy$ in terms of p and q.

17 If $\log_4 x = p$, show that $\log_2 x = 2p$. Hence find (a) the value of k if $\log_4 k = 2 + \log_2 k$ and (b) the value of n if $\log_2 n + \log_4 n = 9$.

18 Solve the equations

(a) $2 \log_5 x = \log_5 (2x + 3)$
(b) $3 \log_2 x = \log_2 (3x - 2)$
(c) $\log_3 (x^2 + 2) = 1 + \log_3 (x + 2)$
(d) $\log_4 (x^2 + 8x - 1) = 2 + \log_4 (x - 1)$
(e) $\log_2 (2x^2 + 3x + 5) = 3 + \log_2 (x + 1)$
(f) $\log_4 (x + 17) = 2 \log_4 (x - 3)$
(g) $\log_2 (x^2 - x + 2) = 1 + 2 \log_2 x$
(h) $\log_5 x = 1 - \log_5 (x - 4)$

19 (a) If $\log_a b = x$, deduce that $x \log_b a = 1$ and hence show that $\log_a b = \frac{1}{\log_b a}$.
(b) Find $\log_2 8$ and hence state the value of $\log_8 2$.

20 If $\log_a x$, $\log_a y$ and $\log_a z$ are three consecutive terms of an AP, show that x, y and z are consecutive terms of a GP.

Common Logarithms

For practical calculation, base 10 is used and logarithms on this base are called **common logarithms.** These are written as **lg** x, which is an abbreviation for $\log_{10} x$. 10 is chosen as it is the base of the decimal system of numbers.

To see the advantage of base 10, suppose we know that lg 5.6 = 0.748. Then lg 560 = $\lg 5.6 \times 10^2 = \lg 5.6 + 2 \lg 10 = 0.748 + 2 = 2.748$ (as lg 10 = 1). The decimal part .748 is unchanged. Similarly lg 5600 would be 3.748. On any other base the logarithms of these numbers would be quite different.

Tables of common logarithms are available but they can be found directly using the LOG (or LG) key on a calculator.

There is another system of logarithms, called **natural logarithms,** written as **ln** x, which is used in Calculus. The base of natural logarithms is a certain number e (≈ 2.718). We shall see the reason for this in Chapter 18.

Logarithmic Equations

Example 19

Find $\log_2 7$.

If $\log_2 7 = x$, then $2^x = 7$.

We cannot express 7 as a power of 2 directly so we convert this equation to a **logarithmic equation** using logarithms of base 10.

Take the lg of each side.

Then $\lg 2^x = \lg 7$.

Hence $x \lg 2 = \lg 7$ and so $x = \frac{\lg 7}{\lg 2}$

$= 2.81$ (by calculator correct to 3 sig. figs.)

Hence $\log_2 7 = 2.81$.

(Verify the result by using the x^y function on the calculator).

Example 20

Find x if $3^{x-1} = 2^{x+1}$.

Convert to a logarithmic equation. Take the lg of each side.

Then $\lg 3^{x-1} = \lg 2^{x+1}$ i.e. $(x - 1) \lg 3 = (x + 1) \lg 2$.

Now solve for x. We do not find lg 2, lg 3 yet.

$x \lg 3 - \lg 3 = x \lg 2 + \lg 2$ which gives $x(\lg 3 - \lg 2) = \lg 3 + \lg 2$

i.e. $x \lg \frac{3}{2} = \lg 6$ or $x = \frac{\lg 6}{\lg 1.5} = 4.42$ by calculator.

(*Note:* The right hand side is NOT $\lg \frac{6}{1.5}$).

Example 21

Find x if $\log_x 6 = 1.5$.

If $\log_x 6 = 1.5$, then $x^{1.5} = 6$.

Taking the lg of each side,

$1.5 \lg x = \lg 6$ and $\lg x = \frac{\lg 6}{1.5} = 0.5188$ (by calculator).

Hence $x = 10^{0.5188} = 3.30$ by calculator, using the x^y function.

Example 22

In Example 19, Chapter 13 we found the least value of n where $0.9^{n-1} < 0.4$. This can also be done using logarithms.

$\lg 0.9^{n-1} < \lg 0.4$ i.e. $(n - 1) \lg 0.9 < \lg 0.4$.

However we **cannot** now write $n - 1 < \frac{\lg 0.4}{\lg 0.9}$ as $\log 0.9 < 0$ and so the inequality sign must be reversed. (Division by a negative quantity).

So $n - 1 > \frac{\lg 0.4}{\lg 0.9} = 8.696$ by calculator

and hence $n > 9.696$ and we take the integral value $n = 10$.

Note: To solve an inequality, if an integral result is required, the calculator method (using x^y) as in Chapter 13, is very suitable and quick. However to solve an *equation* such as $0.9^x = 0.4$, the logarithmic method must be used. Here $x = 8.70$ (to 3 significant figures).

Example 25

Show that $\log_a b = \frac{\lg b}{\lg a}$. *Hence find the value of* $\log_2 5 \times \log_{11} 4 \times \log_5 11$.

Let $\log_a b = x$. Then $a^x = b$. Now take the lg of each side and

$$x \lg a = \lg b \text{ so } x = \frac{\lg b}{\lg a}.$$

Using this result,

$$\log_2 5 \times \log_{11} 4 \times \log_5 11 = \frac{\lg 5}{\lg 2} \times \frac{\lg 4}{\lg 11} \times \frac{\lg 11}{\lg 5}$$

$$= \frac{\lg 4}{\lg 2} = \frac{\lg 2^2}{\lg 2} = \frac{2 \lg 2}{\lg 2} = 2$$

Exercise 15.6 *(Answers on page 639.)*

Give answers correct to 3 significant figures if not exact.

1 Find the value of x if

(a) $3^x = 5$ (b) $2^{x-1} = 7$ (c) $6^{x+1} = 8$
(d) $5^{2x+1} = 3^{2-x}$ (e) $2^{2x+1} = 3^{1-x}$ (f) $2^x = 1.5$
(g) $4^{x-1} = 7$ (h) $5^{3x+2} = 7^{6x-1}$ (i) $2^{2x-1} = 3^{2-x}$
(j) $1.3^x = 5$ (k) $0.6^x = 0.4$ (l) $0.8^{x-1} = 0.2^x$

2 Calculate $\log_3 5$ and $\log_5 7$.

3 If $\log_x 3 = 17$, find the value of x.

4 What is the least number of terms of the GP 3, 4, $\frac{16}{3}$, ... that can be added for their sum to be greater than 90?

5 If the sum of n terms of the GP 8, 12, 18,... is not to exceed 500, what is the largest value of n?

6 Find the least integral value of x if (a) $1.8^{x-1} > 47$, (b) $0.75^x < 0.15$.

7 In how many years will $3000 invested at 5% per year compound interest amount to $5000?

8 After how many years will $9000 amount to $20 000 if it is invested at 4.5% per year compound interest?

9 Given that $P = 50(0.75)^n$, find (a) the value of P when $n = 4$, (b) the value of n when $P = 10$.

10 The population of a city in 1980 was 3 200 000 and this was an increase of 1.7% over the population in 1979. If this rate of increase is continued, in what year will the population first exceed 5 000 000?

11 The decay of a radioactive substance is given by the formula $M = M_0 e^{-0.2t}$ where M_0 is the initial mass, M the mass after t years and e = 2.718. Calculate the half-life of the substance, i.e. the number of years taken for the mass to be halved.

12 Find the value of $\log_3 49 \times \log_5 9 \times \log_7 5$.

13 Find the value of $\log_5 9 \times \log_3 7 \times \log_7 2 \times \log_2 25$.

14 If $\log_5 8 \times \log_2 x = 3$, find the value of x.

15 Given that $\lg y = 1 - 3 \lg x$, show that y can be expressed in the form $y = px^q$ and find the values of p and q.

16 Show that $\lg \frac{1}{10} = -1$. Hence find the values of x which satisfy the equation $\lg(\sin x) + 1 = 0$ for $0° < x < 180°$.

17 (a) Solve the equation $\log_x 2.5 = 8$.

(b) Find the value of x if $x^{3.2} = 10$.

18 Find x if $\log_x 12 = 5$.

19 $y = ax^b - 2$. Given that $y = 6$ when $x = 2$ and $y = 22$ when $x = 4$, find the values of a and b.

SUMMARY

- Rules for indices: $x^m \times x^n = x^{m+n}$

  $x^m \div x^n = x^{m-n}$

  $(x^m)^n = x^{mn}$

- Negative index: $x^{-n} = \frac{1}{x^n}$

  Zero index: $x^0 = 1$

  Fractional index: $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$

- Exponential form $y = a^x$ $\longleftrightarrow$ $x = \log_a y$ logarithmic form

  $(a > 0)$ a is the base of the logarithm

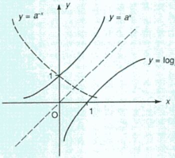

Fig.15.4

13 If $T = T_0e^{-0.5t}$, show that $t = 2\left(\frac{\lg \frac{T_0}{T}}{\lg e}\right)$.

Hence find the value of t given that $e = 2.72$, $T_0 = 30$ and $T = 10$.

14 If $\log_3 (x - 6) = 2y$ and $\log_2 (x - 7) = 3y$, show that $x^2 - 13x + 42 = 72^y$. Given that $y = 1$, find the possible value(s) of x.

15 Find the relation between a and b not involving logarithms if $\log_9 a = 2 + \log_3 b$.

16 $2000 is invested at 5% per year compound interest. After how many years will it have amounted to $3500?

17 Inflation in a certain country is 15% per year. If this rate continues unchanged, after how many years will the cost of living have doubled?

18 Draw the graph of $y = 2^x$ for $0 \le x \le 3$ taking values of x at intervals of 0.5. By adding a suitable straight line to your graph, find an approximate solution of the equation $2^{x+1} + x = 4$.

19 Sketch the graphs of $y = \lg x$ and $y = \lg 10x$. State the coordinates of the points where each curve meets the x-axis.

20 (a) Draw the graph of $y = 2^x$ for $0 \le x \le 2$ taking scales of 2 cm for 1 unit on each axis. Add the line $y = x$ and hence draw the graph of $y = \log_2 x$ for $1 \le x \le 4$.
(b) Calculate the value of $\log_2 6$.
(c) Express $x2^x = 6$ in the form $\log_2 x = px + q$ stating the values of p and q.
(d) What is the equation of the straight line that must be added to the graph to find the solution of the equation $x2^x = 6$?
(e) Draw this line and hence solve the equation approximately.

21 (a) Solve the equations (i) $2 \times 4^{x+1} = 16^{2x}$, (ii) $\log_2 y^2 = 4 + \log_2 (y + 5)$.
(b) Given that $y = ax^n + 3$, that $y = 4.4$ when $x = 10$ and $y = 12.8$ when $x = 100$, find the values of n and of a. (C)

22 Solve the equation $\lg (\cos^2 x) + 2 = 0$ for $0° \le x \le 360°$.

23 Show that the sequence $\lg k$, $\lg 10k$, $\lg 100k$,... forms an AP and find the sum of the first 10 terms of this AP.

B

24 Solve the simultaneous equations $\log_2 x - \log_4 y = 4$, $\log_2 (x - 2y) = 5$.

25 Solve the simultaneous equations $9^x = 27^y$, $64^{xy} = 512^{x+1}$.

26 If $\log_3 2 = a$ and $\log_3 13 = b$, express $\log_{78} 52$ in terms of a and b.

27 Solve the inequality $\log_2 (\log_3 x) > 0$.

28 Given that $\log_8 (p + 2) + \log_8 q = r - \frac{1}{3}$ and that $\log_2 (p - 2) - \log_2 q = 2r + 1$, show that $p^2 = 4 + 32^r$. If $r = 1$, find the possible values of p and q.