

Remainder and Factor Theorems: Cubic Equations

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THE REMAINDER THEOREM

Consider the polynomial $x^3 - 2x^2 + 4x - 3$. Divide this by $x - 3$. Using long division as in Arithmetic, the steps are as follows.

$$\begin{array}{r} \textcircled{1} \quad x^2 + x + 7 \\ x - 3 \overline{) x^3 - 2x^2 + 4x - 3} \\ \textcircled{2} \quad \underline{x^3 - 3x^2} \\ \textcircled{3} \quad x^2 + 4x \quad \textcircled{4} \\ \textcircled{5} \quad \underline{x^2 - 3x} \\ \quad \textcircled{6} \quad 7x - 3 \quad \textcircled{7} \\ \quad \textcircled{8} \quad \underline{7x - 21} \\ \quad \textcircled{9} \quad 18 \text{ remainder} \end{array}$$

Step $\textcircled{1}$ Divide x^3 by x to give x^2

Step $\textcircled{2}$ Multiply $x - 3$ by x^2

Step $\textcircled{3}$ Subtract to get x^2 and bring down the next term, $4x$

Step $\textcircled{4}$ Divide x^2 by x to give x

Step $\textcircled{5}$ Multiply $x - 3$ by x

Step $\textcircled{6}$ Subtract to get $7x$ and bring down the next term, -3

Step $\textcircled{7}$ Divide $7x$ by x to give 7

Step $\textcircled{8}$ Multiply $x - 3$ by 7

Step $\textcircled{9}$ Subtract; this gives the remainder 18

In the following it is the remainder which is important.

Let $f(x) = x^3 - 2x^2 + 4x - 3$. Now find $f(3)$. What do you notice?

This result is not a coincidence. In fact we shall prove that when a polynomial $f(x)$ is divided by a linear expression $x - a$ the remainder will be $f(a)$.

If we divide say 15 by 4, then the result (the quotient) is 3 and the remainder is 3. The connection between these is

$$15 = \underset{\substack{\uparrow \\ \text{divisor}}}{4} \times \underset{\substack{\uparrow \\ \text{quotient}}}{3} + \underset{\substack{\uparrow \\ \text{remainder}}}{3}$$

So if we divide $f(x)$ by $(x - a)$ and the quotient is Q , and the remainder R , then

$$f(x) = (x - a) \times Q + R$$

This is true for all values of a .

Now put $x = a$.

$$\text{Then } f(a) = 0 \times Q + R$$

$$\text{i.e. } R = f(a).$$

If we divide by the general linear expression $px + q$, then $f(x) = (px + q) \times Q + R$.

$$\text{Put } x = -\frac{q}{p} \text{ and } f\left(-\frac{q}{p}\right) = R.$$

This is the **remainder theorem** for a polynomial $f(x)$:

$$\text{If } f(x) \text{ is divided by } px + q, \text{ the remainder is } f\left(-\frac{q}{p}\right)$$

It is also worth remembering the simple form: when $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$.

The theorem only applies to *polynomials* and only to **linear** divisors. Note also that it tells us nothing about the quotient.

Example 1

What are the remainders when $x^2 - x^2 + 3x - 2$ is divided by (a) $x - 1$, (b) $x + 2$, (c) $2x - 1$?

(a) Here, $x - a$ is $x - 1$ so $a = 1$.

$$f(1) = 1 - 1 + 3 - 2 = 1$$

The remainder is 1.

(b) Here, $a = -2$.

$$f(-2) = -8 - 4 - 6 - 2 = -20$$

The remainder is -20.

(c) $px + q = 2x - 1$, so $-\frac{q}{p} = \frac{1}{2}$.

$$f\left(\frac{1}{2}\right) = \frac{1}{8} - \frac{1}{4} + \frac{3}{2} - 2 = -\frac{5}{8}$$

The remainder is $-\frac{5}{8}$.

Example 2

The polynomial $x^3 + ax^2 - 3x + 4$ is divided by $x - 2$ and the remainder is 14. What is the value of a ?

Taking $f(x)$ as the given expression,

the remainder = $f(2) = 8 + 4a - 6 + 4 = 6 + 4a$.

Then $6 + 4a = 14$ and $a = 2$.

Example 3

$f(x) = x^3 + ax^2 + bx - 3$. When $f(x)$ is divided by $x - 1$ and $x + 1$, the remainders are 1 and -9 respectively. Find the values of a and b .

Dividing by $x - 1$, the remainder = $f(1) = 1 + a + b - 3 = a + b - 2$.

Then $a + b - 2 = 1$, so $a + b = 3$.

Dividing by $x + 1$, the remainder is $f(-1) = -1 + a - b - 3 = a - b - 4$.

Then $a - b - 4 = -9$, so $a - b = -5$.

Solving the two equations for a and b , $a = -1$ and $b = 4$.

Example 4

The polynomial $f(x) = A(x - 1)^2 + B(x + 2)^2$ is divided by $x + 1$ and $x - 2$. The remainders are 3 and -15 respectively. Find the values of A and B .

Divisor $x + 1$: remainder = $f(-1) = A(-1 - 1)^2 + B(-1 + 2)^2 = 4A + B = 3$

Divisor $x - 2$: remainder = $f(2) = A(2 - 1)^2 + B(2 + 2)^2 = A + 16B = -15$

Solving the simultaneous equations, $A = 1$ and $B = -1$.

Example 5

(i) If the expression $x^3 + px^2 + qx + r$ gives the same remainder when divided by $x + 1$ or $x - 2$, show that $p + q = -3$.

(ii) If the remainder is 4 when the expression is divided by $x - 1$, find the value of r .

(iii) If also the remainder is -60 when the expression is divided by $x + 3$, find the values of p and q .

(i) Divisor $x + 1$: remainder = $f(-1) = -1 + p - q + r$

Divisor $x - 2$: remainder = $f(2) = 8 + 4p + 2q + r$

Then $-1 + p - q + r = 8 + 4p + 2q + r$

so $3p + 3q = -9$ or $p + q = -3$.

(ii) Divisor $x - 1$: remainder = $f(1) = 1 + p + q + r = 4$.

But $p + q = -3$

Then $1 - 3 + r = 4$ and $r = 6$.

(iii) Divisor $x + 3$: remainder = $f(-3) = -27 + 9p - 3q + 6 = -60$

Hence $9p - 3q = -39$ or $3p - q = -13$.

Solving the simultaneous equations in p and q , $p = -4$, $q = 1$.

Exercise 12.1 (Answers on page 634.)

1 Find the remainder when $x^3 + 2x^2 - x - 1$ is divided by

- (a) $x - 1$ (b) $x + 1$ (c) $x - 3$ (d) $x + 4$
(e) $2x - 1$ (f) $3x + 2$ (g) $2x - 3$ (h) $x + t$

2 Find the remainder when

- (a) $x^2 - 7x - 3$ is divided by $x + 2$
(b) $x^4 - 3x^2 - x + 3$ is divided by $x + 3$
(c) $3x^3 - x^2 - x - 1$ is divided by $x - 4$

3 Find the remainder when the following expressions are divided by the linear expression stated:

- (a) $1 - 2x - 3x^2$ by $x - 2$ (b) $x^3 + 3x - 3$ by $3x - 2$
(c) $x^3 + x^2 - x - 3$ by $x + 3$ (d) $3x^3 - x^2 + 4$ by $3x + 1$
(e) $2x^3 - x^2 + 4x + 2$ by $2x + 1$ (f) $x^4 + x^3 - 2x^2 - 3$ by $x + 3$

4 If $2x^3 - x^2 - 1$ is divided by $x + 2$, what is the remainder?

5 If $x^3 - 2x + 1$ is divided by $3x + 2$, what is the remainder?

6 What is the remainder when $ax^3 + bx^2 + cx + d$ is divided by $x + 1$?

7 The expression $2x^3 + px^2 - x - 2$ is divided by $x + 3$. State the remainder in terms of p .

8 Given $f(x) = ax^3 + x^2 - 3x - 2$ and that the remainder on dividing $f(x)$ by $x + 2$ is 0, what is the value of a ?

9 $x^3 + px - 4$ is divided by $x + 4$ and the remainder is -28 . Find the value of p .

10 The polynomial $x^3 + ax^2 + bx - 1$ is divided by $x - 2$ and $x + 1$.
The remainders are 7 and 4 respectively. Find the value of a and of b .

11 When the expression $x^3 + px^2 + qx + 2$ is divided by $x + 2$, the remainder is double that obtained when it is divided by $x - 1$. Find a relation between p and q . If the remainder is also 6 when the expression is divided by $x + 1$, find the value of p and of q .

12 The remainders obtained when $px^3 + qx^2 + 4x - 2$ is divided by $x - 1$ and $x + 2$ are equal. Show that $3p - q = -4$. If also the remainder is -18 when the polynomial is divided by $x - 2$, find the value of p and of q .

- 13 The expression $x^2 - 4x - 2$ has the same remainder when it is divided by either $x - a$ or $x - b$ ($a \neq b$). Show that $a + b = 4$.
Given also that the remainder is 10 when the expression is divided by $x - 2a$, find the values of a and b .
- 14 When $x^3 - x^2 + ax + b$ is divided by $x - 1$ and $x + 1$, the remainders are -5 and -1 respectively. Find the values of a and b .
- 15 The polynomial $x^4 + 3x^3 + ax^2 + bx - 1$ is divided by $x - 1$ and $x + 2$.
The remainders obtained are 4 and 19 respectively. Find the values of a and b .
- 16 When the polynomials $x^3 - 4x + 3$ and $x^3 - x^2 + x + 9$ are each divided by $x - a$, the remainders are equal. Find the possible values of a .
- 17 If $x^2 + (m - 2)x - m^2 - 3m + 5$ is divided by $x + m$, the remainder is -1 . Find the values of m .

THE FACTOR THEOREM

If $(x - a)$ is a factor of $f(x)$, then there will be **no** remainder when $f(x)$ is divided by $(x - a)$. So $f(a) = 0$. Similarly, if $px + q$ is a factor of $f(x)$, $f\left(-\frac{q}{p}\right) = 0$. This is the **factor theorem** for a polynomial $f(x)$:

If $px + q$ is a factor of $f(x)$, $f\left(-\frac{q}{p}\right) = 0$.

If $f\left(-\frac{q}{p}\right) = 0$, $px + q$ is a factor of $f(x)$.

We use the factor theorem to factorize polynomials (if possible).

Example 6

Factorize $x^3 - 6x^2 - x + 6$.

Take $f(x)$ as $x^3 - 6x^2 - x + 6$. As $f(x)$ is of the third degree, it will have *at most* three linear factors of the form $px + a$, $qx + b$, $rx + c$.

$$\text{Then } x^3 - 6x^2 - x + 6 = \underbrace{(px + a)(qx + b)(rx + c)}$$

As the first term is x^3 , $p = q = r = 1$.

$$\text{So } x^3 - 6x^2 - x + 6 = \underbrace{(x + a)(x + b)(x + c)}$$

The last term is $+6$ so $a \times b \times c = +6$.

Hence the possible factors come from $x \pm 1$, $x \pm 2$, $x \pm 3$, $x \pm 6$.

The first factor has to be found by trial.

Try $x - 1$. Then $f(1) = 1 - 6 - 1 + 6 = 0$ so $x - 1$ is a factor.

Now we could continue trying other possible factors. In simple cases, this would be quick and satisfactory but in general could be time consuming, especially if the polynomial had only one linear factor.

For a cubic polynomial the best method is to find one factor by trial and then deduce the remaining quadratic factor by inspection. The steps are shown in full here but in practice all the working is done mentally and only the results written down.

$$x^3 - 6x^2 - x + 6 = (x - 1)(\quad).$$

We now complete the blank bracket step by step.

$$x^3 - 6x^2 - x + 6 = (x - 1)(x^2 + tx - 6)$$

Step ① The first term in the second bracket must be x^2

Step ② The last term must be -6 as $-1 \times -6 = +6$

Step ③ Take the middle term as tx . To find t , equate the coefficients of x^2 .

$$\underbrace{-6x^2}_{\text{from } (x-1)(x^2+tx-6)} = \underbrace{-x^2}_{\text{from } (x-1)x^2} + tx^2$$

$$\text{So } -6x^2 = -x^2 + tx^2 \text{ and } t = -5.$$

Check the coefficient of x : $(x - 1)(x^2 - 5x - 6)$ i.e. $+5x - 6x = -x$ which is correct.

$$\text{Hence } f(x) = (x - 1)(x^2 - 5x - 6) = (x - 1)(x - 6)(x + 1).$$

Example 7

Factorize $x^3 - 3x^2 - 2x + 8$.

The possible factors are $x \pm 1$, $x \pm 2$, $x \pm 4$, $x \pm 8$. The first factor is found by trial.

Try $x + 1$: remainder = $-1 - 3 + 2 + 8 \neq 0$. $x + 1$ is not a factor.

Try $x - 1$: remainder = $1 - 3 - 2 + 8 \neq 0$. $x - 1$ is not a factor.

Try $x + 2$: remainder = $-8 - 12 + 4 + 8 \neq 0$. $x + 2$ is not a factor.

Try $x - 2$: remainder = $8 - 12 - 4 + 8 = 0$. $x - 2$ is a factor.

$$\text{Then } x^3 - 3x^2 - 2x + 8 = (x - 2)(x^2 - x - 4)$$

Hence the expression = $(x - 2)(x^2 - x - 4)$ as $x^2 - x - 4$ cannot be factorized.

Example 8

Factorize $4x^3 - 8x^2 - x + 2$.

The first term $4x^3$ makes the solution more complicated as one of the factors must be $2x + a$ or $4x + b$. However try $x - 1$ and $x + 1$ and confirm that they are not factors.

Now try $x - 2$: remainder = $32 - 32 - 2 + 2 = 0$ so $x - 2$ is a factor.

$$\begin{aligned} \text{Then } 4x^3 - 8x^2 - x + 2 &= (x - 2)(4x^2 \quad - 1) \\ &\quad \underbrace{\hspace{1.5cm}}_{=-8x^2} \quad \uparrow \text{no middle term} \end{aligned}$$

Hence the expression = $(x - 2)(4x^2 - 1) = (x - 2)(2x - 1)(2x + 1)$

Example 9

The expression $2x^3 + ax^2 + bx - 2$ is exactly divisible by $x - 2$ and $2x + 1$. Find the values of a and b and hence find the third factor.

Divide by $x - 2$: remainder = $16 + 4a + 2b - 2 = 0$ so $4a + 2b = -14$ or $2a + b = -7$.

$$\begin{aligned} \text{Divide by } 2x + 1: \text{ remainder} &= 2\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) - 2 \\ &= -\frac{1}{4} + \frac{a}{4} - \frac{b}{2} - 2 = 0 \text{ so } a - 2b = 9. \end{aligned}$$

Solving the two equations for a and b , $a = -1$, $b = -5$.

Let $(px + q)$ be the third factor.

$$\begin{aligned} \text{Hence } 2x^3 - x^2 - 5x - 2 &= (2x + 1)(x - 2)(px + q) \\ &= (2x^2 - 3x - 2)(px + q) \end{aligned}$$

By inspection, $p = 1$ and by checking the last term, $q = +1$.

Hence the third factor is $x + 1$.

Exercise 12.2 (Answers on page 634.)

1 Factorize

- | | |
|-----------------------------|-----------------------------|
| (a) $x^2 + 1$ | (b) $x^3 - 4x^2 + 5x - 2$ |
| (c) $x^2 - 4x^2 + x + 6$ | (d) $x^3 + 6x^2 + 11x + 6$ |
| (e) $x^3 - x^2 + 2x - 2$ | (f) $x^3 + 3x^2 - 6x - 8$ |
| (g) $2x^3 + 7x^2 + 8x + 3$ | (h) $3x^3 + 2x^2 - 3x - 2$ |
| (i) $x^2 - 1$ | (j) $x^3 - 2x^2 - 9x + 18$ |
| (k) $2x^3 - 3x^2 - 8x + 12$ | (l) $6x^3 - 13x^2 + 9x - 2$ |

2 If $x^3 + ax + 6$ is divided by $x + 1$, the remainder is 12. Find the value of a and factorize the expression.

3 Given the expression $ax^3 + bx^2 + cx + d$, show that $x - 1$ is a factor if $a + b + c + d = 0$. [This result is worth remembering: if the sum of the coefficients = 0, then $x - 1$ is a factor.]

- 4 The expression $2x^3 + ax^2 + bx + 1$ is exactly divisible by $2x - 1$ and $x + 1$. Find the value of a and of b and hence find the third factor of the expression.
- 5 Given that $f(x) = x^3 - 6x^2 + 11x + p$, find the value of p for which $x - 3$ is a factor of $f(x)$. With this value of p , find the other factors of $f(x)$.
- 6 $x^3 + ax + b$ and $x^2 + 2ax + 3b$ have a common factor $x + 1$. Find the value of a and of b and with these values factorize the cubic expression.
- 7 $f(x) = x^3 + ax^2 + bx + 4$ and $f(x)$ is exactly divisible by $x - 2$. If the remainder is -24 when $f(x)$ is divided by $x + 2$, find the value of a and of b and hence factorize $f(x)$.
- 8 $x^3 + ax^2 + x + b$ is exactly divisible by $x - 3$ and the remainder is -20 when it is divided by $x + 2$. Find the values of a and b and then factorize the expression.
- 9 The function $f(x) = 2x^3 + ax^2 - 2x + b$ has a factor $2x - 1$. When $f(x)$ is divided by $x + 2$, the remainder is -15 . Find the value of a and of b and find the other two factors of $f(x)$.
- 10 Given that $f(x) = x^3 + ax^2 + bx + 8$ and that the remainders when $f(x)$ is divided by $x + 1$ and $x + 2$ are 6 and -8 respectively, find the value of a and of b and hence factorize $f(x)$.
- 11 If $x - 2$ is a common factor of the expressions $x^2 + (p + q)x - q$ and $2x^2 + (p - 1)x + (p + 2q)$, find the value of p and of q .
- 12 The remainder when $x(x + b)(x - 2b)$ is divided by $x - b$ is -16 . Find the value of b .
- 13 Find the value of k ($\neq 0$) for which $x + k$ and $x - k$ are both factors of $x^3 - x^2 - 9x + 9$. Then find the third factor.
- 14 The expression $x^4 + 4x^3 + 6x^2 + 5x + 2$ has only two linear factors. Find these factors.
- 15 The expression $A(x - 1)^3 + B(x + 3)^2 + 20$ is exactly divisible by $x + 1$ and the remainder is 26 when it is divided by x . Find the value of A and of B . Using these values, rewrite the expression as a polynomial and factorize completely.

Solving a Cubic Equation

Example 10

Solve the equation $2x^3 + 3x^2 - 3x = 2$.

First we factorize the polynomial $2x^3 + 3x^2 - 3x - 2$.

The sum of the coefficients is 0 so $x - 1$ is a factor.

Then the polynomial is $(x - 1)(2x^2 + 5x + 2) = (x - 1)(2x + 1)(x + 2)$.

Hence the roots of the cubic equation $2x^3 + 3x^2 - 3x - 2 = 0$ are $x = 1$ or $-\frac{1}{2}$ or -2 .

Example 11

If $f(x) = x^3 - 3x^2 + x + 2$, solve the equation $f(x) = 0$.

$x - 1$ is not a factor of $f(x)$.

Verify that $x + 1$ is also not a factor.

Try $x - 2$ and verify that this is a factor.

Then $f(x) = (x - 2)(x^2 - x - 1)$.

The roots of $f(x) = 0$ are $x = 2$ and the roots of $x^2 - x - 1 = 0$,

i.e. $x = \frac{1 \pm \sqrt{5}}{2} = 1.62$ or -0.62 .

Example 12

Given that $f(x) = x^3 - 2x^2 + 2x$, solve the equation $f(x) = 4$.

$f(x) = 4$ gives $x^3 - 2x^2 + 2x - 4 = 0$. To solve this equation, we first factorise the polynomial $x^3 - 2x^2 + 2x - 4$.

Check that $x + 1$ and $x - 1$ are not factors. Now try $x - 2$.

Divide the polynomial by $x - 2$ to obtain the other factor.

The equation is $(x - 2)(x^2 + 2) = 0$ and the only root is $x = 2$ as $x^2 + 2 = 0$ has no real roots.

Example 13

Find the nature and x -coordinates of the turning points on the curve

$$y = 3x^4 + 4x^3 - 6x^2 - 12x + 1.$$

$$\frac{dy}{dx} = 12x^3 + 12x^2 - 12x - 12 = 12(x^3 + x^2 - x - 1)$$

$$\frac{dy}{dx} = 0 \text{ when } x^3 + x^2 - x - 1 = 0.$$




$x - 1$ is a factor of the left hand side of this equation.

Then $x^3 + x^2 - x - 1 = (x - 1)(x^2 + 2x + 1) = (x - 1)(x + 1)^2$ and so $\frac{dy}{dx} = 0$ when $x = 1$ or $x = -1$.

$$\frac{d^2y}{dx^2} = 12(3x^2 + 2x - 1)$$

When $x = 1$, $\frac{d^2y}{dx^2} > 0$ so this is a minimum point.

When $x = -1$, $\frac{d^2y}{dx^2} = 0$ so we use the sign test on $\frac{dy}{dx} = (x - 1)(x + 1)^2$.

x	slightly < -1	-1	slightly > -1
sign of $\frac{dy}{dx}$	-	0	-
sketch of tangent			

At $x = -1$, the curve has a point of inflexion.

IDENTICAL POLYNOMIALS

If we state that two cubic polynomials are **identical**, then corresponding coefficients must be equal.

So if $2x^3 - x^2 + x - 5 = ax^3 + bx^2 + cx + d$ then $a = 2$, $b = -1$, $c = 1$ and $d = -5$.

We can also say that the polynomials are equal for **all** values of x .

This enables us to convert a polynomial into a form which may be more suitable for further computations. The method is general and applies to any polynomials of the same degree.

Example 14

Given that $2x^3 - x^2 - 7x - 5 = (Ax + B)(x - 1)(x + 2) + C$ for all values of x , evaluate A , B and C .

First expand the right hand side, obtaining

$$\begin{aligned}(Ax + B)(x^2 + x - 2) + C &= Ax^3 + (B + A)x^2 + (B - 2A)x - 2B + C \\ &= 2x^3 - x^2 - 7x - 5\end{aligned}$$

Now compare coefficients:

The x^3 term gives $A = 2$.

The x^2 term gives $B + A = -1$ so $B = -3$.

Check that the x -coefficients are equal. $B - 2A = -3 - 4 = -7$ which is correct. finally $-2B + C = -5$ so $C = -11$.

An alternative method is to substitute suitable values of x into each polynomial, remembering that these are equal for all values of x .

Put $x = 1$. Then $2 - 1 - 7 - 5 = 0 + C$ so $C = -11$. Note why 1 was chosen. What other value of x could we have chosen instead?

Now put $x = 0$. Then $-5 = (B)(-1)(2) - 11$ so $B = -3$.

Put $x = -1$. Then $-2 - 1 + 7 - 5 = (-A - 3)(-2)(1) - 11$

i.e. $-1 = 2A + 6 - 11$ so $A = 2$.

Either method is simple to use, but the second method needs a careful choice of values for x .

Exercise 12.3 (Answers on page 634.)

1 Solve the equations

- | | |
|--------------------------------|---------------------------------|
| (a) $x^3 + x^2 - x = 1$ | (b) $x^3 + 2x^2 - x = 2$ |
| (c) $x^3 + 6x^2 + 11x + 6 = 0$ | (d) $x^3 - 4x^2 + 5x - 2 = 0$ |
| (e) $x^3 + 2x^2 - 2x + 3 = 0$ | (f) $x^3 = 13x + 12$ |
| (g) $x^3 - 9x^2 + 26x = 24$ | (h) $x^3 = 6x + 5$ |
| (i) $x^3 - 4x^2 + x + 6 = 0$ | (j) $4x^3 - 12x^2 + 5x + 6 = 0$ |
| (k) $x^2(2x + 1) = 13x - 6$ | |

2 The expression $x^3 + ax^2 + bx + 12$ is exactly divisible by $x - 1$ and $x + 3$. Find the value of a and of b and the remaining factor of the expression. Hence solve the equation $x^3 + ax^2 + bx + 12 = 0$.

3 $(x - 2)$ is a factor of $2x^3 + ax^2 + bx - 2$ and when this expression is divided by $x + 3$, the remainder is -50 . Find the value of a and of b and the other factors. Hence solve the equation $2x^3 + ax^2 + bx = 2$.

4 In each of the following, the polynomials are equal for all values of x . Evaluate A, B and C.

- (a) $x^3 - x^2 - 2x - 7 = x(Ax + B)(x + 1) + C$
(b) $3x^3 - 8x^2 + 4x - 5 = x(Ax + B)(x - 2) + C$
(c) $3x^3 - 13x^2 + 18x - 10 = (Ax + B)(x - 1)(x - 2) + C$
(d) $4x^3 - 7x^2 - 5x + 6 = (Ax + B)(x - 2)(x + 1) + C$
(e) $2x^2 - x + 3 = A(x + 1)^2 + B(x + 1) + C$
(f) $2x^3 - 7x^2 + 7x - 5 = A(x - 1)^3 + Bx(x - 1) + C$

5 (a) Solve the equation $x^3 - 7x + 6 = 0$. Hence state the solutions of the equation $(x - 2)^3 - 7(x - 2) + 6 = 0$.

(b) Solve the equation $2x^3 = 11x^2 - 17x + 6$.

6 If $f(x) = x^3 + ax^2 + bx + 6$ and the remainders when $f(x)$ is divided by $x + 1$ and $x - 2$ are 20 and 8 respectively, find the value of a and of b and hence solve the equation $f(x) = 0$.

7 (a) $f(x) = x^3 + ax^2 + bx + 12$. Given that the remainders when $f(x)$ is divided by $x + 1$ and $x + 3$ are 12 and -30 respectively, find the value of a and of b . With these values, solve the equation $f(x) = 0$.

(b) $f(x) = x^3 + kx^2 + 3x - 2$ is divided by $x + k$. If the remainder is 4, find the value divisible by $x - 4$. With this value for k , solve the equation $f(x) = 4(x - 1)$.

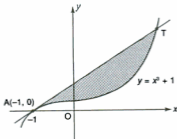
8 Find the x -coordinates of the points where the line $y = 5x - 1$ meets the curve $y = 2x^3 + x^2 + 1$.

9 Find the coordinates of the points of intersection of the curve $y = x^3$ and the line $y = 7x + 6$.

10 Show that $2x^3 - x^2 + 3x - 4$ cannot be equal to $x(Ax + B)(x - 2) + C$ for all values of x . State the new coefficient of x in the first polynomial which will make the polynomials identical.

- 11 Find the x -coordinates of the points where the line $y = x + 6$ meets the curve $y = x^3 + 3x^2 + x + 2$ and show that the line is a tangent to the curve at one of these points.
- 12 Find the x -coordinates and the type of the turning points on the curve $y = x^4 - 8x^3 + 22x^2 - 24x + 4$.
- 13 Fig. 12.1 shows part of the curve $y = x^3 + 1$. The tangent at A $(-1, 0)$ meets the curve again at T. Find
- the equation of AT,
 - the coordinates of T,
 - the area of the shaded region in the figure.

Fig. 12.1



- 14 The tangent at P $(1, 1)$ on the curve $y = x^3$ meets the curve again at Q. Find
- the equation of PQ,
 - the coordinates of Q,
 - the area of the finite region enclosed by the tangent and the curve.

SUMMARY

- Remainder theorem:** if a polynomial $f(x)$ is divided by $px + q$, the remainder is $f(-\frac{q}{p})$; if divided by $x - a$, the remainder is $f(a)$.
- Factor theorem:** if $px + q$ is a factor of $f(x)$, $f(-\frac{q}{p}) = 0$;
if $x - a$ is a factor of $f(x)$, $f(a) = 0$.
- If $f(-\frac{q}{p}) = 0$, $px + q$ is a factor of $f(x)$;
if $f(a) = 0$, $x - a$ is a factor of $f(x)$.
- If the sum of the coefficients of $f(x)$ is 0, then $x - 1$ is a factor of $f(x)$.

REVISION EXERCISE 12 (Answers on page 635.)

A

- 1 The expression $ax^3 - x^2 + bx - 1$ leaves remainders of -33 and 77 when divided by $x + 2$ and $x - 3$ respectively. Find the value of a and of b and the remainder when divided by $x - 2$.
- 2 (a) The expression $6x^2 + x + 7$ leaves the same remainder when divided by $x - a$ and $x + 2a$, where $a \neq 0$. Calculate the value of a .
(b) Given that $x^2 + px + q$ and $3x^2 + q$ have a common factor $x - b$, where p, q and b are non-zero, show that $3p^2 + 4q = 0$. (C)
- 3 (a) Find, in terms of p , the remainder when $3x^3 - 2x^2 + px - 6$ is divided by $x + 2$. Hence write down the value of p for which the expression is exactly divisible by $x + 2$.
(b) Solve the equation $x^3 - 12x + 16 = 0$.
(c) Given that the expression $x^3 + ax^2 + bx + c$ leaves the same remainder when divided by $x - 1$ or $x + 2$, prove that $a = b + 3$.
Given also that the remainder is 3 when the expression is divided by $x + 1$, calculate the value of c . (C)
- 4 Find the x -coordinates of the points where the curves $y = x^3 - 4x^2 - 5$ and $y = 2x^2 - 11x + 1$ intersect.
- 5 (a) The expression $2x^3 + ax^2 - 72x - 18$ leaves a remainder of 17 when divided by $x + 5$. Determine the value of a .
(b) Solve the equation $2x^3 = x^2 + 5x + 2$.
(c) Given that the expression $x^2 - 5x + 7$ leaves the same remainder whether divided by $x - b$ or $x - c$, where $b \neq c$, show that $b + c = 5$.
Given further that $4bc = 21$ and that $b > c$, find the value of b and of c . (C)
- 6 (a) Given that $x + 2$ is a factor of $f(x) = x^3 - 3x^2 - 4x + p$ find the value of p and hence factorise $f(x)$.
(b) Solve the equation $2x^3 + 3x^2 - 4x - 4$, giving the answers correct to 2 decimal places if necessary.
(c) If $4x^3 - 11x^2 - 6x + 7 = (Ax + B)(x + 1)(x - 3) + C$ for all values of x , evaluate A , B and C .
- 7 (a) The expressions $x^3 - 7x + 6$ and $x^3 - x^2 - 4x + 24$ have the same remainder when divided by $x + p$.
(i) Find the possible values of p .
(ii) Determine whether, for either or both of these values, $x + p$ is a factor of the expressions.
(b) Given that $E = x^4 - x^3 + 5x^2 + 4x - 36$, find (i) the remainder when E is divided by $x + 1$, (ii) the value of a ($a > 0$) for which $x + a$ and $x - a$ are both factors of E . (C)

- 8 If the polynomials (i) $3x^3 + x^2 + 2x + 4$ and (ii) $x^3 + 2x^2 + 6x - 10$ are each divided by $x - a$, the remainder from (i) is double the remainder from (ii). Find the possible values of a .
- 9 The gradient of a curve at the point (x, y) is given by $3x^2 - 12x + 12$ and the curve passes through the point $(0, -7)$. Find the equation of the curve and the coordinates of the point(s) where it meets the x -axis.
- 10 If the vectors $\mathbf{r}_1 = (t - 1)\mathbf{i} - (t + 2)\mathbf{j}$ and $\mathbf{r}_2 = t^2\mathbf{i} + (t - 1)\mathbf{j}$ are perpendicular, find the possible values of t .
- 11 (a) Find the remainder when $x^3 + 3x - 2$ is divided by $x + 2$.
 (b) Find the value of a for which $(1 - 2a)x^2 + 5ax + (a - 1)(a - 8)$ is divisible by $x - 2$ but not by $x - 1$.
 (c) Given that $16x^4 - 4x^3 - 4b^2x^2 + 7bx + 18$ is divisible by $2x + b$,
 (i) show that $b^3 - 7b^2 + 36 = 0$,
 (ii) find the possible values of b . (C)
- 12 The tangent at the point $P(1, 3)$ on the curve $y = x^3 - x + 3$ meets the curve again at T . Find (i) the equation of PT , (ii) the coordinates of T and (iii) the area of the finite region enclosed by the curve and the tangent.
- 13 Fig. 12.2 (not drawn to scale) shows parts of the curves $y = x^2 + 2$ and $y = x(2x + 1)$. Find the coordinates of the points A , B and C where the curves intersect and the areas of the shaded regions P and Q .

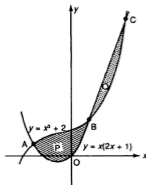


Fig. 12.2

- 14 By first solving the equation $x^2 - 3x + 2 = 0$ or otherwise, find the solutions of the equation $(x + 2)^2 = 3x + 4$.

B

- 15 If $f(x) = x^3 - 3x^2 + x - 3$, show that $x - 2$ is a factor of $f(x + 1)$.
- 16 Given that $2x + p$ is a factor of $2x^4 + px^3 + 2px^2 + 7x + 3$, show that $p^3 - 7p + 6 = 0$ and hence find the possible values of p .

- 17 Solve the equation $6 \cos^3 \theta = 7 \cos^2 \theta - 1$ for $0^\circ \leq \theta \leq 360^\circ$.
- 18 Factorize the polynomial $f(x) = x^3 - (2p + 1)x^2 + (2p - q)x + q$ where p and q are constants. If the equation $f(x) = 0$ has three real roots, show that $p^2 + q \geq 0$.
- 19 If the polynomial $ax^3 + bx^2 + cx - 4$ is divided by $x + 2$, the remainder is double that obtained when the polynomial is divided by $x + 1$. Show that c can have any value and find b in terms of a .
- 20 If the solutions of the equation $x^3 + px^2 + qx + r = 0$ are a , b and c it can be proved that $a^2 + b^2 + c^2 = p^2 - 2pq$ and that $a^3 + b^3 + c^3 = 3pq - p^3 - 3r$.
Verify these statements for the equation $x^3 + 2x^2 - 5x - 6 = 0$.
Make up a cubic equation $(x - a)(x - b)(x - c) = 0$ with values for a , b and c and verify the statements for your equation.
- 21 Find the ranges of values of x for which
 $2x^3 - 3x^2 + x + 6 < (x + 2)(x + 1)(x - 1)$